Emergent Yang-Mills Theory

University of the Witwatersrand
Author: Shaun de Carvalho, Student number: 542425
Supervisor: Professor Robert de Mello Koch

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Declaration

I declare that this Dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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(Signature of candidate)

__________ 26th __________ day of __________ January __________ 20 ________17________ in __________ Johannesburg __________

Abstract

In this dissertation we tackle the question: is there an emergent Yang-Mills theory coming from the low energy description of branes and open strings? This new Yang-Mills theory has no connection to the original gauge symmetry of the CFT. We thus explore a large N but non-planar limit of the theory. This is done with new methods developed in group representation theory. A study the dilatation operator D in $\mathcal{N} = 4$ SYM theory is done since its eigenvalue, the anomalous dimension, is mapped to the energy of the open string in the IIB string theory. The construction of the spherical harmonics from the harmonic expansion on the 3-sphere, S^3, is done to understand the theory of the giant graviton's worldvolume. The light-front parton picture is examined, since it explains how one can "glue" single momentum modes together to obtain higher momentum modes, and we believe that this procedure is described dynamically using magnon bound states. Following from this, we work on determining the exact magnon bound state spectrum. Finally, we test our hypothesis and see if the spectrum of the bound states matches the harmonic spectrum from the harmonic expansion on the 3-sphere, S^3. A non-trivial check is also performed to show that the bound state spectrum does indeed match the spectrum coming from $\mathcal{N} = 4$ SYM.

Acknowledgements

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Contents

1 Introduction

It is apparent that a duality exists between conformal field theory (CFT) and string theories. The duality implies a one-to-one correspondence between the states in the Hilbert space of the CFT and the states in the Hilbert space of the quantum gravity. Further, thanks to the state-operator correspondence, we should thus have a local operator in the CFT for each state in string theory. Membranes and their excitations, provided by open strings, are states that appear in the Hilbert space of quantum gravity as provided by string theory. These states have many characteristic features: at low energy they should be described by a Yang-Mills theory and this includes things like the Gauss Law, that will have a highly non-trivial effect on the spectrum of the theory. Thus, a new Yang-Mills theory emerges from the branes and their open string excitations. We say this Yang-Mills theory emerges, because it has nothing to do with the original gauge symmetry of the CFT and it lives on a new space that emerges form the field theory at strong coupling. The dual descriptions of branes and open strings in CFT are known. From string theory we know that super Yang-Mills (SYM) describes open string dynamics at low energy. We are exploring a large N but non-planar limit of the theory. In this case, the usual large N methods are not useful and new methods need to be employed - this is why group representation theory plays a role; it provides the new methods we need. The question we would like to tackle is: is there an emergent Yang-Mills theory coming from the low energy description of branes and open strings?

There are results that motivate this study. Namely, when you study the one-loop mixing problem in the conformal field theory (which is dual to a problem of computing energies of states), one finds that the operators that have a good scaling dimension perfectly match the states of a brane plus open string excitations Hilbert space. Further, in this picture the Gauss Law of the gauge theory that should emerge from the branes is clearly visible. Since this is a major motivation for our study, we start by reviewing this. We now come to the statement that the energy of the open string in the IIB string theory is mapped to the anomalous dimension, which is the eigenvalue of dilatation operator D, in $\mathcal{N}=4$ SYM theory. We will now motivate this identification. When working in $\mathbb{R}^4$ Euclidean space, we consider the metric

$$ds^2 = dt^2 + d\vec{x}\cdot d\vec{x} = dr^2 + r^2 d\Omega_3^2.$$

Scaling is a symmetry of the dynamics since we study a CFT. Now, let $r = e^\tau$ such that

$$ds^2 = e^{2\tau} d\tau^2 + e^{2\tau} d\Omega_3^2,$$

which is related by a conformal transformation to

$$ds^2 = d\tau^2 + d\Omega_3^2,$$

which is $\mathbb{R}\times S^3$. Performing the scaling $r \to \lambda r = e^a r,\ \lambda \neq 0$, we see that

$$\begin{aligned} e^\tau &\to e^a e^\tau = e^{a+\tau} \\ &\Rightarrow \tau \to \tau + a. \end{aligned}$$

From this insight, we see that the dilatation operator starts to look like a Hamiltonian since it generates τ translations. However, since we are doing a radial quantisation (see figures 1 and 2 that illustrates this), translations *do not* map from a Hilbert space back to a Hilbert space. This implies $e^{-ia^\mu P_\mu}$ is *not* a unitary operator (i.e. $[e^{-ia^\mu P_\mu}]^\dagger \neq e^{-ia^\mu P_\mu}$), where P_μ generates translations, and finally this implies $P_\mu^\dagger \neq P_\mu$.

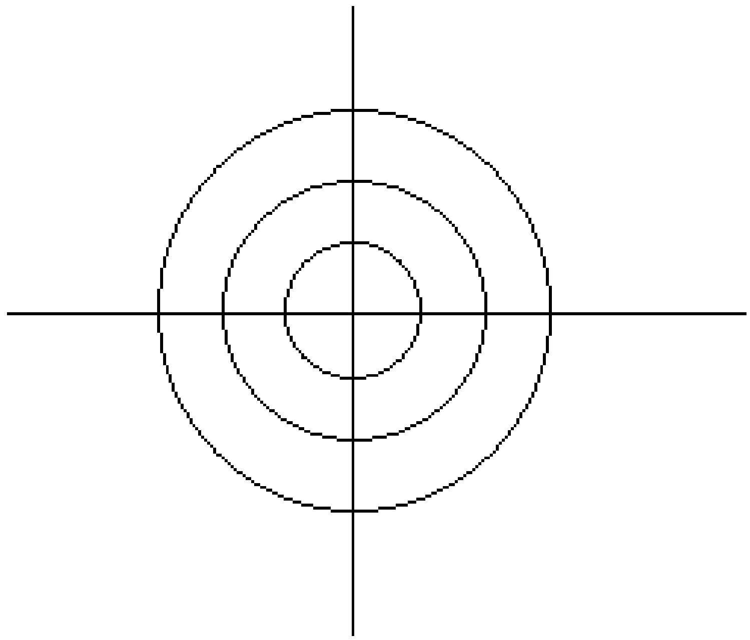


Figure 1: In this figure radial quantisation at a fixed radius is used and each ring represents a Hilbert space.

Usually when we have a hermitian operator, we have the condition $A^\dagger(0) = A(0)$, where the time evolution of operator $A(0)$ is given as $A(t) = e^{iHt}A(0)e^{-iHt}$, and thus $(A(t))^\dagger = e^{iHt}A^\dagger(0)e^{-iHt} = e^{iHt}A(0)e^{-iHt} = A(t)$. However, we will now introduce a new dagger operation such that $A^\ddagger(0) = A(0)$, where $A(\tau) = e^{H\tau}A(0)e^{-H\tau}$ and $it \to \tau$. We introduce this new dagger operation so that we can represent a relationship between the generators P_μ and K_μ, which is the special conformal generator. However, note that

$$A^\ddagger(\tau) = e^{-H\tau}A(0)e^{H\tau} \neq e^{H\tau}A(0)e^{-H\tau} = A(\tau),$$

but instead

$$A^\ddagger(\tau) = e^{H(-\tau)}A(0)e^{-H(-\tau)} = A(-\tau),$$

which is reflection positivity. Introduce the inversion map $I : x^\mu \to \frac{x^\mu}{x\cdot x}$. Applying the new dagger operator on a state we get

$$(|\psi\rangle)^\ddagger = \langle\psi|I$$

and on an operator we get

$$\mathcal{O}^\ddagger = I\mathcal{O}^\dagger I,$$

and $I\mathcal{O}^\dagger I = I\mathcal{O}I$ if the operator is hermitian. Therefore, we see that

$$P_\mu^\ddagger = IP_\mu^\dagger I = IP_\mu I = K_\mu.$$

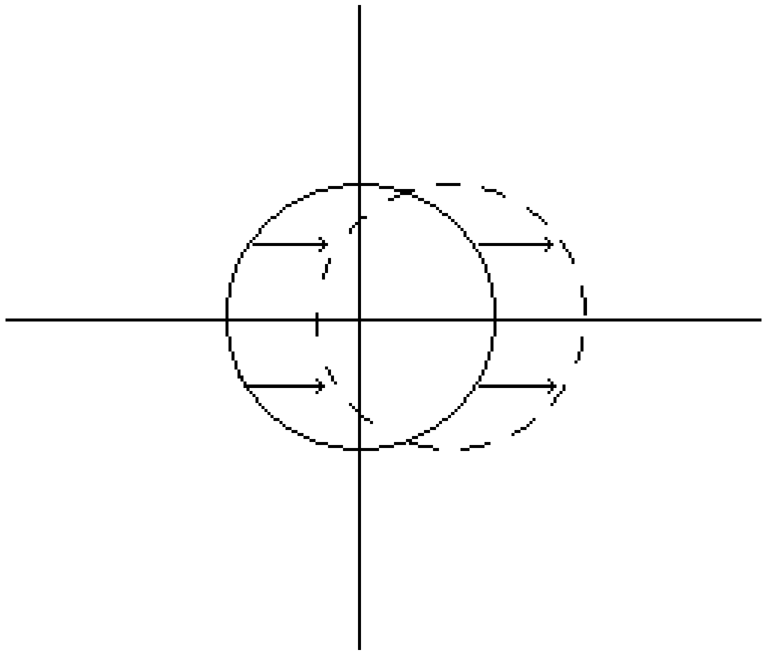
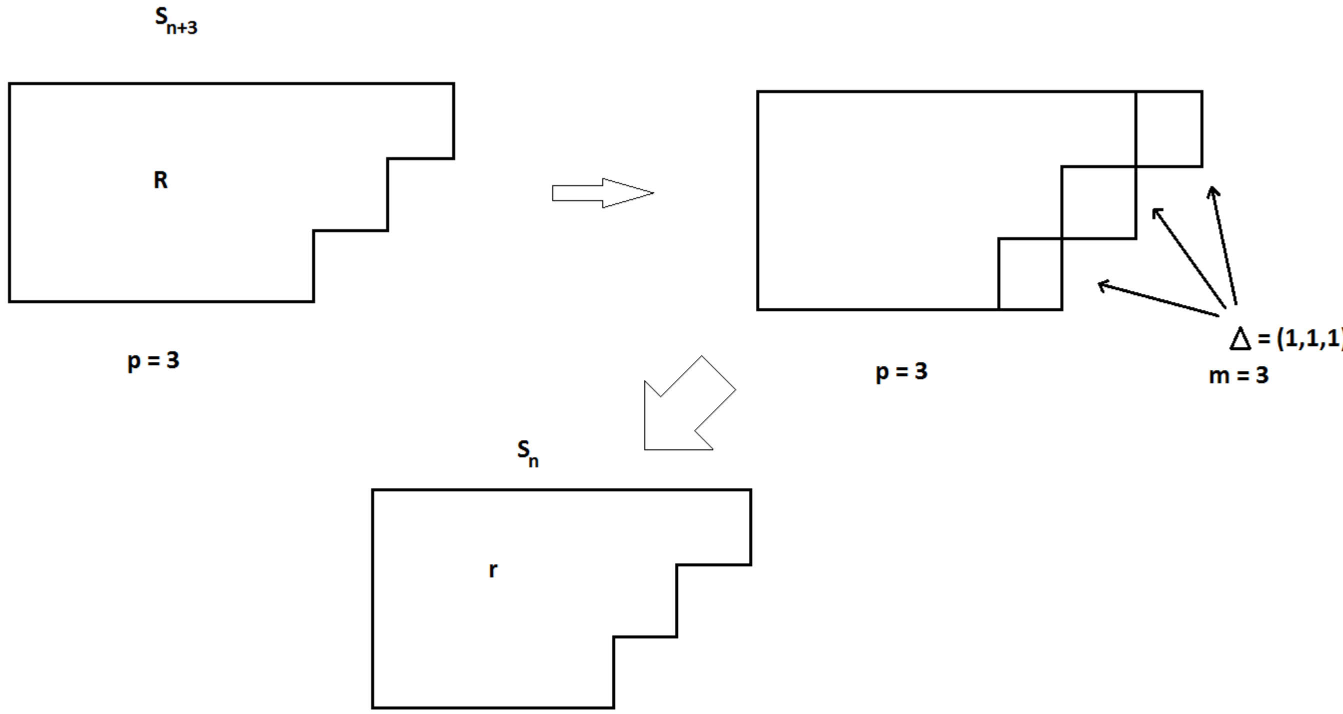


Figure 2: In this figure we see that by translating a Hilbert space, we intersect with other Hilbert spaces, therefore translations do not map from a Hilbert space back to a Hilbert space.

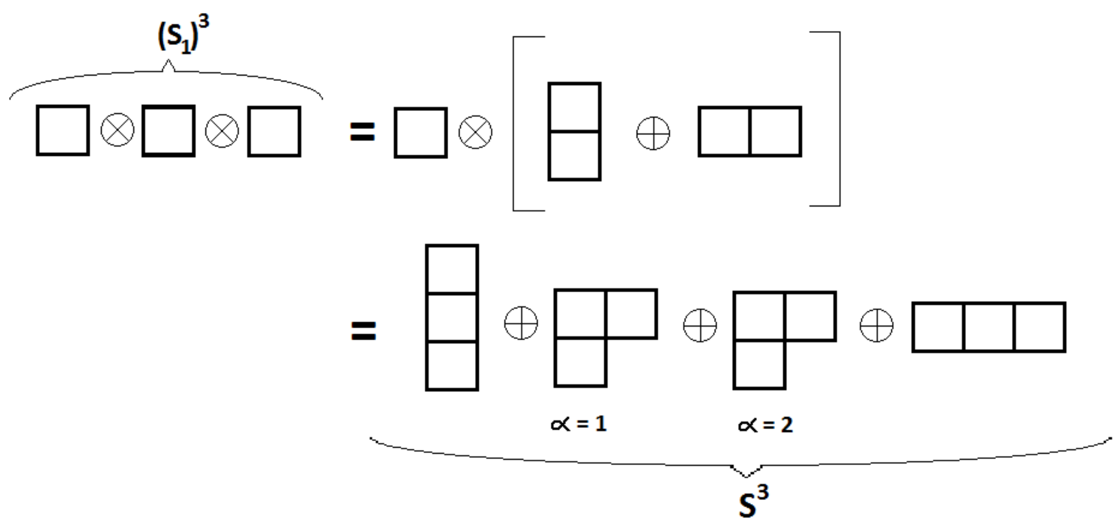
From this, we see that P_μ and K_μ, through the new dagger operator, act as the creation and annihilation operators of the dilatation operator D the same way $a^\dagger$ and a are the creation and annihilation operators of the Hamiltonian H. The last thing to note is that $D^\ddagger = IDI = -D$, i.e. it is anti-hermitian.

We will be studying threebranes carrying a D3 dipole charge, known as giant gravitons which are $D3$ branes with a spherical worldvolume, S^3. They are stable due to the fact that they couple to the background RR five form flux and carry a non-zero angular momentum, leading to a Lorentz-type force which causes them to puff out and expand. This Lorentz force exactly cancels the force due to their tension which tries to collapse them. When accelerating a giant graviton it grows and the faster you move it, the bigger it gets. There is a limit on how big a giant graviton can get. This limit arises because the giant graviton cannot expand to a size that is bigger than S^5. This cut off on the size of the giant translates into a cut off on the angular momentum of the giant. When giant graviton states are excited, they can be described in terms of open strings which end on the brane. For p coincident giant gravitons we expect a $U(p)$ gauge theory. We have some good ideas about what the local operators that are dual to giant gravitons are. We will introduce these operators as they provide a basis for the local operators of the theory. We will also be solving the problem of diagonalising the dilatation operator in this basis. Working in the language of Young-Tableaux, suppose we have a Young diagram R with $n+m$ boxes that labels an irreducible representation (irrep) of S_{n+m} that has p rows and $p = O(1)$. Every row has $O(N)$ boxes and the difference between the length of adjacent rows is $O(N)$. We can construct operators called *restricted Schur polynomials* [1], [2] denoted as $\chi_{R,(r,s)\alpha\beta}$ where labelled irrep r of S_n is the Young diagram with n boxes that we have once m boxes are removed from Young diagram R to form Young diagrams s which labels an irrep of S_m, and α and β are multiplicity labels that label copies of (r, s).

Now suppose, for example, that $p = 3$ and $m = 3$, i.e. we remove 3 boxes from Young diagram R which labels an irrep of S_{n+3}. Using the Δ weight $\Delta = (1, 1, 1)$, which removes one box from each row, we are left with a Young diagram r that labels an irrep of S_n and we have the irrep of $(S_1)^3$ to compose Young diagrams s that labels irreps of S_3, i.e. $(S_1)^3 \to S_3$.



There are 4 valid Young diagrams that can be constructed from the product of the 3 single boxes where two of the diagrams are the same, and thus $\alpha = 1, 2$ (it is the same for β). Therefore there are 6 possible restricted Schur polynomials that can be made; two from the non-repeating diagrams and four from the repeating diagram.



We will also work with an operator Δ_{ij} because it plays a role in the diagonalisation of the dilatation operator. When Δ_{ij} acts on a Young diagram R, what results is the product of the sum of the content of the box at the end of the i^{th} row and the content[1] of the box at the end of the j^{th} row (i.e. $c_i + c_j$) with the Young diagram R along with the product of the negative square root of the product of contents c_i and c_j (i.e. $-\sqrt{c_i c_j}$) with the Young diagram R where the box at the end of the i^{th} row is moved to the end of the j^{th} row and with the Young diagram R where the box at the end of the j^{th} row is moved to the end of the i^{th} row. Note that this operator can also operate on Young diagram r. As an example, acting with the operator Δ_{13} on Young diagram $\ydiagram{3,2,1}$, we get

$$
\begin{aligned}
\Delta_{13}\ydiagram{3,2,1} &= (c_1 + c_3)\ydiagram{3,2,1} - \sqrt{c_1 c_3}\left[\ydiagram{2,2,2} + \ydiagram{4,2}\right] \\
&= (2 + (-2))\ydiagram{3,2,1} - \sqrt{2(-2)}\left[\ydiagram{2,2,2} + \ydiagram{4,2}\right] \\
&= -2i\left[\ydiagram{2,2,2} + \ydiagram{4,2}\right].
\end{aligned}
$$

Since the physics of these CFT operators is the physics of branes and open strings in string theory, we expect that, in a suitable limit, the dynamics of the Gauss graph operators will reproduce the physics of a Yang-Mills theory. Gaining insight into how this actually happens is a big motivation for the present study. The simplest test we can do is to ask if the spectra coming from the CFT operators matches the spectra expected from Yang-Mills theory. For the case of a single brane, we expect to get electromagnetism on a 3-sphere S^3. When considering the first Maxwell's equation, i.e. Gauss's law, on the membrane worldvolume (on S^3), we have

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}.$$

Integrating on both sides over the volume of the 3-sphere, we get

$$\int_{S^3} \vec{\nabla} \cdot \vec{E}\ d^3\theta = \int_{S^3} \frac{\rho}{\varepsilon_0} d^3\theta.$$

Note that the right hand side is equal to the total charge on the 3-sphere divided by the permittivity of free space, i.e. $\frac{Q_{tot}}{\varepsilon_0}$. On the left hand side we can use the divergence theorem to obtain

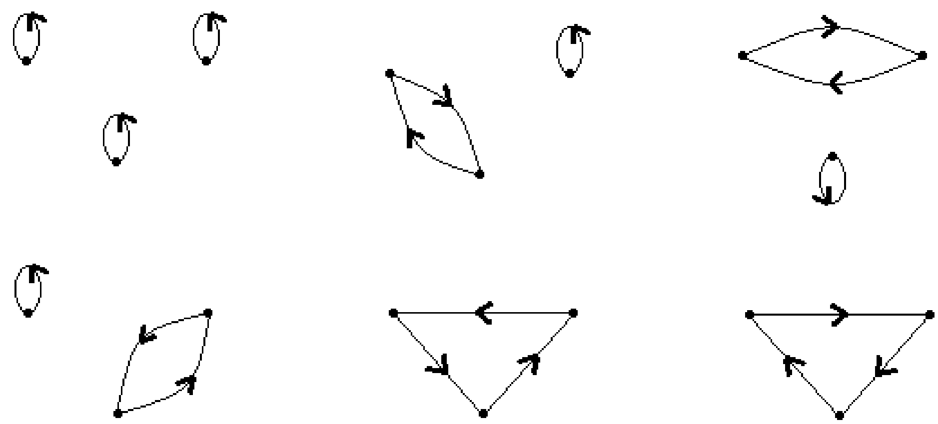
$$\int_{S^3} \vec{\nabla} \cdot \vec{E}\ d^3\theta = \int_{\partial S^3} \vec{E} \cdot d\vec{A} = \int_{S^3} \frac{\rho}{\varepsilon_0} d^3\theta = \frac{Q_{tot}}{\varepsilon_0}.$$

Open strings are oriented: an endpoint can be described as "incoming" or "outgoing". It should be further stated that incoming and outgoing endpoints have opposite charge. Note that the second integral above is equal to zero since $\partial S^3 = 0$. This then demonstrates that the total charge on a membrane is zero. Since the endpoints of open strings are charged this implies that the number of outgoing open strings on a membrane equal to the number of ingoing open strings on a membrane.

We are now describing the result of diagonalising the dilatation operator. So going back to the example where Young diagram R labels the irrep of S_{n+3} which lead us to the construction of 6 restricted Schur polynomials, we can associate these constructions to Gauss graphs where each graph has 3 branes and 3 open strings (this is due to there being 3 rows in the example), where the size of the membrane(s) corresponds to the row length(s) in the Young diagram, i.e. the longest

[1]The content of a box in row i and column j is $N + j - i$.

row in the Young diagram corresponds to the largest membrane and the shortest row in the Young diagram corresponds to the smallest membrane. The number of boxes in a row is equal to the number of fields and each field adds an R-charge of 1. Thus, number of boxes in a row = amount of R-charge = amount of angular momentum. However, more angular momentum means more Lorentz force and hence a bigger size. So a large number of boxes means a big brane. The way the open strings are attached to the membranes follow the 6 separate permutations namely $\mathbb{1}$, (1 3), (1 2), (2 3), (1 3 2), and (1 2 3); this can be explicitly seen in the diagram below:



In each permutation case we see that the number of incoming and outgoing open strings is the same for each dot (brane) thus resulting in a zero overall charge for each dot. The Gauss Law constraint states that the total charge on the giant's world volume must vanish. Therefore, these diagrams realise the Gauss Law since they obey its constraint. We also introduce the parameter n_{ij} where if $n_{ij} = 1$, then it means an open string is connected from brane i to brane j, and if $n_{ij} = 0$ then it means an open string is *not* connected from brane i to brane j. Note that if $n_{ij} = 1$, it does not mean $n_{ji} = 1$; n_{ij} follows strictly "from i to j" (the same is true for when $n_{ij} = 0$). With this background, we can now state the action of the dilatation operator D on normalised operators $O(\sigma_k)$ (which is discussed in greater detail in section 2.7). The action is given as

$$DO(\sigma_k) = \sum_{i \neq j} n_{ij} \Delta_{ij} O(\sigma_k).$$

This dissertation is organised as follows: in section 2 we introduce the essential tools that focus on the large N, non-planar limits of CFT along with methods used for studying matrix models. A large amount of the methods and tools are presented in the language of Group and Representation Theory of finite groups. These tools will be used to help us when we tackle how to show how a Yang-Mills theory can emerge from CFT operators. In section 3 we explore the spectrum of the worldvolume of a giant graviton, which is a 3-sphere. We use techniques from group theory to perform expansions. This is a standard part of harmonic analysis. We do all this to construct the spherical harmonics of the coset $SO(4)/SO(3)$ which is demonstrated to be the 3-sphere, S^3. Checks will be carried out to verify the validity of the harmonics constructed. In section 4 we review the work of A. Jevicki [3] on light-front partons and its association with dimensional reduction – which is seen in the example of relativistic field theory. Through a number of considerations and manipulations, a relativistic field can be expressed in terms of non-relativistic partons of a lower dimensional theory, a new understanding into the features of local field theory is given. In section 5 we begin with the discussion as to why poles are associated with bound states. We then review known work on magnons and the magnon S-matrix which leads to us determining the exact magnon bound state spectrum. We also review the work of N. Dorey [4] on magnon bound states as we identify magnons as the basic partons and magnon bound states with the coincident patron configurations since it is highly important in determining the spectrum of magnon bound states. In section 6 we compare the results determined in section 3, with the harmonics constructed, with the results in section 4 and 5 of the spectrum of the magnon bound states, and see if they match. And lastly, in section 7 we will review what was done throughout this dissertation and what was concluded in section 6. The main idea used to achieve our goal was first done by V. Balasubramanian, D. Berenstein, B. Feng, M. Huang in [5]

2 Review of Useful Tools

2.1 Introduction

In this chapter we lay out the tools necessary to explore large N but non-planar limits of the CFT. The new methods are most easily developed by studying matrix models. We will therefore introduce matrix models and consider their large N but non-planar limits. A lot of the tools are expressed in the language of Group Theory and Representation Theory of finite groups. We will motivate in detail why matrix models are string theories. We consider the matrix model with a single field Z, the matrix model with two fields Z and Y, and then generalise to matrix models with many fields. Finally, we will introduce the one loop dilatation operator and its action on normalised operators that are proportional to restricted Schur polynomials [1] [2]. Many of the concepts presented in this chapter is a review of existing work. However, we delve into these concepts for the sake of setting up the problem and using the tools developed here in assisting us in tackling the demonstration of how Yang-Mills theory can emerge from a class of CFT operators.

2.2 Matrix Models

Gaussian integrals are good toy models for the path integral. In the same way, matrix models are good models for QCD (gauge theory). When studying a hermitian matrix, i.e. $M = M^\dagger$, $M_{ij} = M^*_{ji}$, where $i, j = 1, \dots , N$, it should be noted that:

- The diagonal elements are in total N real numbers.
- The off-diagonal elements are complex, *but* elements below the diagonal are complex conjugates (c.c.) of elements above the diagonal.
- The number of elements above the diagonal is equal to $(N^2 - N)/2$, so that there are $N^2 - N$ off-diagonal real numbers.

In conclusion, there are N^2 real numbers in total: real diagonal + real off diagonal $= N + N^2 - N = N^2$. The model is defined as follows: any expectation value is given by evaluating the integral

$$\langle \dots \rangle = \int [dM] e^{-\frac{\omega}{2} \operatorname{Tr} M^2} \dots$$

where $M_{ii}, M_{ij} = M^r_{ij} + iM^i_{ij}$ if $i > j$ and $M_{ij} = M^*_{ij}$ if $i < j$. Thus the measure for the integral is

$$\int [dM] \cdots = \mathcal{N} \prod_{i=1}^{N} \int_{-\infty}^{\infty} dM_{ii} \prod_{k,l=1}^{N} \prod_{k>l} \int_{-\infty}^{\infty} dM^r_{kl} \int_{-\infty}^{\infty} dM^i_{kl} \dots$$

Note that this is N^2 real integrals. $\mathcal{N}$ sets the normalisation of the measure. We chose $\mathcal{N}$ so that $\int [dM] e^{-\frac{\omega}{2} \operatorname{Tr} M^2} = \langle 1 \rangle = 1$.

For general N:

$$\mathcal{N}\int_{-\infty}^{\infty} dM_{11}e^{-\frac{\omega}{2}M_{11}^2}\dots\int_{-\infty}^{\infty} dM_{NN}e^{-\frac{\omega}{2}M_{NN}^2}\int_{-\infty}^{\infty} dM_{N1}^i e^{-\omega M_{N1}^{i^2}}\int_{-\infty}^{\infty} dM_{N1}^r e^{-\omega M_{N1}^{r^2}}\dots$$
$$\dots\int_{-\infty}^{\infty} dM_{N(N-1)}^i e^{-\omega M_{N(N-1)}^{i^2}}\int_{-\infty}^{\infty} dM_{N(N-1)}^r e^{-\omega M_{N(N-1)}^{r^2}} = 1$$
$$\Rightarrow \mathcal{N}\left(\sqrt{\frac{2\pi}{\omega}}\right)^N\left(\sqrt{\frac{\pi}{\omega}}\right)^{N^2-N} = 1$$
$$\Rightarrow \mathcal{N} = \left(\sqrt{\frac{\omega}{2\pi}}\right)^N\left(\sqrt{\frac{\omega}{\pi}}\right)^{N^2-N}$$
$$\Rightarrow \mathcal{N} = \left(\frac{\omega}{2\pi}\right)^{N/2}\left(\frac{\omega}{\pi}\right)^{(N^2-N)/2} = \left(\sqrt{\frac{\omega}{\pi}}\right)^{N^2}\frac{1}{2^{\frac{N}{2}}}.$$

We would like to study correlators of the form

$$\langle M_{ij}M_{kl}\rangle = \int [dM]e^{-\frac{\omega}{2}\operatorname{Tr}M^2}M_{ij}M_{kl}.,$$

Towards this end, consider

$$Z[J] = \int [dM]e^{-\frac{\omega}{2}\operatorname{Tr}M^2+\operatorname{Tr}JM},$$

where $\operatorname{Tr}(JM) = J_{ij}M_{ji}$ (repeated indices are summed). Correlators are given by taking derivatives of $Z[J]$ with respect to the J's. Indeed, noting that

- $\frac{\partial M_{ij}}{\partial M_{kl}} = \delta_{ik}\delta_{jl}$
- $\frac{d}{dM_{ij}}\operatorname{Tr}(JM) = \frac{d}{dM_{ij}}J_{kl}M_{lk} = J_{kl}\delta_{il}\delta_{jk} = J_{ji}$
- $\frac{d}{dJ_{ij}}e^{\operatorname{Tr}(JM)} = e^{\operatorname{Tr}(JM)}\frac{d}{dJ_{ij}}J_{kl}M_{lk} = e^{\operatorname{Tr}(JM)}\delta_{ki}\delta_{lj}M_{lk} = e^{\operatorname{Tr}(JM)}M_{ji}$
- $\frac{d}{dJ_{ij}}\frac{d}{dJ_{kl}}e^{-\frac{\omega}{2}\operatorname{Tr}(M^2)+\operatorname{Tr}(JM)}|_{J=0} = e^{-\frac{\omega}{2}\operatorname{Tr}(M^2)}\frac{d}{dJ_{ij}}\frac{d}{dJ_{kl}}e^{\operatorname{Tr}(JM)}|_{J=0} = e^{-\frac{\omega}{2}\operatorname{Tr}(M^2)}M_{ji}M_{lk},$

we easily find

$$\langle M_{ij}M_{kl}\rangle = \frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}}Z[J]|_{J=0}.$$

This can be generalised to

$$\langle M_{ij}M_{kl}\dots M_{rs}\rangle = \frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}}\dots\frac{d}{dJ_{sr}}Z[J]|_{J=0}.$$

To solve for $Z[J]$, we need to compute a Gaussian integral which amounts to completing the square. Note that $\frac{\omega}{2}\operatorname{Tr}(M^2) - \operatorname{Tr}(JM) = \frac{\omega}{2}\operatorname{Tr}([M-\frac{J}{\omega}]^2) - \frac{1}{2\omega}\operatorname{Tr}(J^2)$. Thus

$$Z[J] = \int [dM]e^{-\frac{\omega}{2}\operatorname{Tr}([M-\frac{J}{\omega}]^2)+\frac{1}{2\omega}\operatorname{Tr}(J^2)}.$$

Changing the integration variables: $M = M' + \frac{J}{\omega} \Rightarrow dM = dM'$ because J is independent of M. Then

$$\begin{aligned} Z[J] &= \int [dM'] e^{-\frac{\omega}{2}\operatorname{Tr}(M'^2) + \frac{1}{2\omega}\operatorname{Tr}(J^2)} \\ &= e^{\frac{1}{2\omega}\operatorname{Tr}(J^2)} \int [dM] e^{-\frac{\omega}{2}\operatorname{Tr}(M^2)} \\ &= e^{\frac{1}{2\omega}\operatorname{Tr}(J^2)}, \end{aligned}$$

since $\int [dM] e^{-\frac{\omega}{2}\operatorname{Tr}(M^2)} = 1$. Now

$$\frac{d}{dJ_{ij}} \operatorname{Tr}(J^2) = \frac{d}{dJ_{ij}} J_{kl}J_{lk} = \delta_{ik}\delta_{lj}J_{lk} + J_{kl}\delta_{il}\delta_{jk} = J_{ji} + J_{ji} = 2J_{ji}.$$

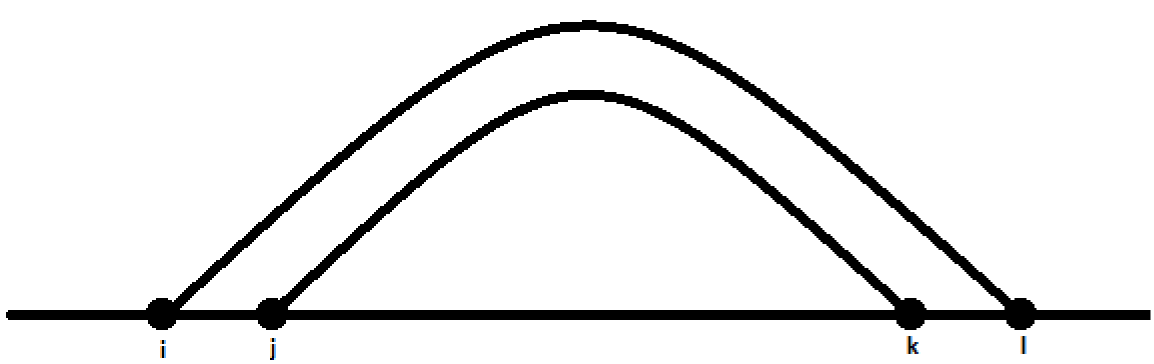
Thus,

$$\begin{aligned} \langle M_{ij} M_{kl} \rangle &= \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} e^{\frac{1}{2\omega}\operatorname{Tr}(J^2)}|_{J=0} \\ &= \frac{d}{dJ_{ji}} \left[\frac{1}{2\omega} e^{\frac{1}{2\omega}\operatorname{Tr}(J^2)} \frac{d}{dJ_{lk}} \operatorname{Tr}(J^2) \right]_{|J=0} \\ &= \frac{d}{dJ_{ji}} \left[\frac{J_{kl}}{\omega} e^{\frac{1}{2\omega}\operatorname{Tr}(J^2)} \right]_{|J=0} \\ &= \left[\frac{\delta_{jk}\delta_{il}}{\omega} + \frac{J_{kl}}{\omega}\frac{J_{ij}}{\omega} \right] e^{\frac{1}{2\omega}\operatorname{Tr}(J^2)}|_{J=0} \\ &= \frac{1}{\omega}\delta_{jk}\delta_{il}. \end{aligned}$$

This is the mathematical expression of the propagator in the matrix model.

2.2.1 Ribbon diagrams

The propagator $\langle M_{ij} M_{kl} \rangle = \frac{1}{\omega}\delta_{jk}\delta_{il}$ can be expressed as a diagram:

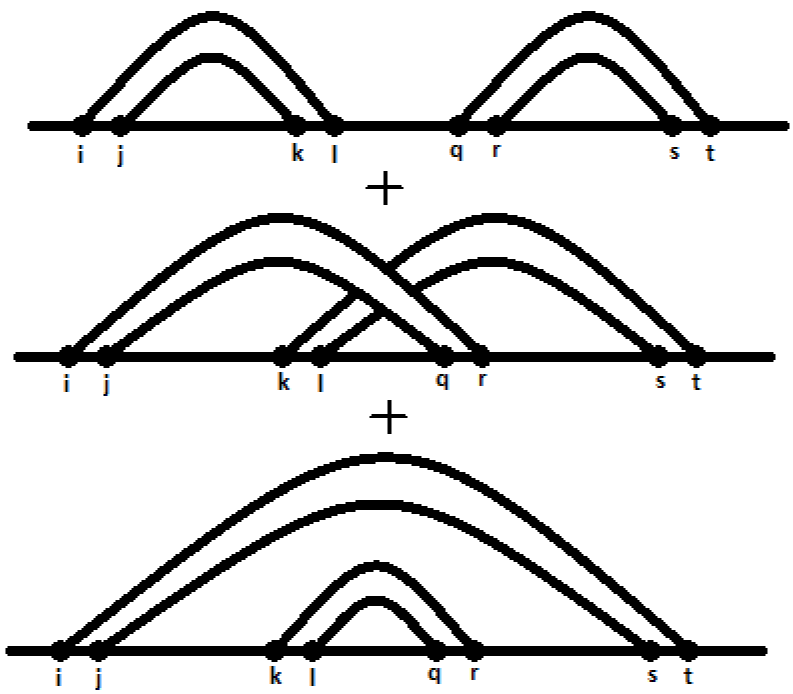


where the indices of each matrix are placed in pairs, in order, on a line. Lines are used to connect the indices of the two matrices in accordance with the indices paired by the Kronecker deltas, i.e. "i" connects to "l" according to the Kronecker delta δ_{il} and "j" connects to "k" according to the Kronecker delta δ_{jk}. Thus an image of a "ribbon" is formed.

The Feynman rules for interpreting ribbon diagrams are as follows:

- Every ribbon comes with a $\frac{1}{\omega}$.
- Every ribbon edge comes with a Kronecker delta.

For example, $\langle M_{ij}M_{kl}M_{qr}M_{st}\rangle$ is represented as the diagrams below:



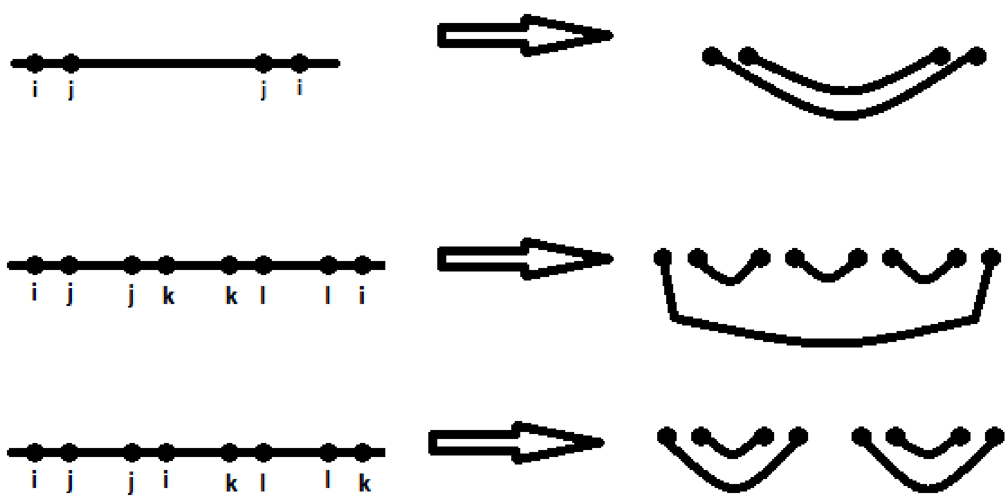
The value of the diagrams (using the Feynman rules) are:

$$\langle M_{ij}M_{kl}M_{qr}M_{st}\rangle = \frac{1}{\omega^2}\delta_{il}\delta_{jk}\delta_{qt}\delta_{rs} + \frac{1}{\omega^2}\delta_{ir}\delta_{jq}\delta_{kt}\delta_{ls} + \frac{1}{\omega^2}\delta_{it}\delta_{js}\delta_{kr}\delta_{lq}.$$

The matrix model is a zero dimensional model. Thus, we can't really sensibly define a local (gauge) symmetry. We simply declare that global $U(N)$ rotations of the hermitian matrix M are a gauge symmetry. In this case, the physical observables are traces of M, since these are the quantities that are naturally gauge invariant.

2.2.2 Improving notation for gauge invariant states

We now simplify our notation. There is no reference to indices in the improved notation. We will connect points that are labelled by the same index since they are summed to give the traces of powers of M_{ij} i.e. they are no free indices. The idea for the diagram for $\langle M_{ij}M_{ji}\rangle$ is illustrated in the diagram below:



In what follows we will make use of the following correlators

- $\langle \mathrm{Tr}(M^2) \rangle = \frac{1}{\omega} N^2$
- $\langle \mathrm{Tr}(M^4) \rangle = \frac{1}{\omega^2}(2N^3 + N)$
- $\langle \mathrm{Tr}(M^2)\, \mathrm{Tr}(M^2) \rangle = \frac{1}{\omega^2}(N^4 + 2N^2)$
- $\langle \mathrm{Tr}(M^2)\, \mathrm{Tr}(M^4) \rangle = \frac{1}{\omega^3}(2N^5 + 9N^3 + 4N)$
- $\langle \mathrm{Tr}(M^4)\, \mathrm{Tr}(M^4) \rangle = \frac{1}{\omega^4}(4N^6 + 40N^4 + 61N^2)$.

Gerard 't Hooft [6] was the first to suggest that it is useful to consider the limit $N \to \infty$. In this limit

- $\langle \mathrm{Tr}(M^2) \rangle = \frac{1}{\omega} N^2$
- $\langle \mathrm{Tr}(M^2) \rangle = \frac{2}{\omega^2} N^3$
- $\langle \mathrm{Tr}(M^2)\, \mathrm{Tr}(M^2) \rangle = \langle \mathrm{Tr}(M^2) \rangle \langle \mathrm{Tr}(M^2) \rangle = \frac{1}{\omega^2} N^4$
- $\langle \mathrm{Tr}(M^2)\, \mathrm{Tr}(M^4) \rangle = \langle \mathrm{Tr}(M^2) \rangle \langle \mathrm{Tr}(M^4) \rangle = \frac{2}{\omega^3} N^5$.

Thus, we learn that

The expectation value of products is equal to the product of expectation values in the $N \to \infty$ limit.

This is a property of the large N theory and it is called *factorisation*.

2.2.3 Factorisation

Our goal now is to understand the physical interpretation and implications of factorisation. Consider a system that can be in a number of different states; these states are labelled with index i. The probability for the system to be in a state i is μ_i. Note that $\sum_i \mu_i = 1,\ \ \mu_i \geq 0\ \forall i$. There is a set of observables O_I and the value of O_I in state i is $O_I(i)$. The expectation value of observables is $\langle O_I \rangle = \sum_i \mu_i O_I(i)$. Factorisation is the statement that

$$
\begin{aligned}
\langle O_{I_1} O_{I_2} \dots O_{I_n} \rangle &= \langle O_{I_1} \rangle \langle O_{I_2} \rangle \dots \langle O_{I_n} \rangle \\
\Rightarrow \sum_i \mu_i O_{I_1}(i) O_{I_2}(i) \dots O_{I_n}(i) &= \sum_{i_1} \mu_{i_1} O_{I_1}(i_1) \sum_{i_2} \mu_{i_2} O_{I_2}(i_2) \cdots \sum_{i_n} \mu_{i_n} O_{I_n}(i_n)
\end{aligned}
$$

where we can consider any O's and n can be anything. Interpreted as a system of equations for the probabilities μ_i, the equations are highly over determined. Fortunately there is a solution and it sets $\mu_i = 1$ for $i = i^*$ and $\mu_i = 0$ for $i \neq i^*$. Then both sides of the equation are equal to $O_{I_1}(i^*) O_{I_2}(i^*) \dots O_{I_n}(i^*)$. The system occupies a definite state, so that the sum over states in the path integral reduces to a single term. Only in the classical limit is the system in a definite state, so that this is a classical limit!

This conclusion holds for any theory of matrix fields so that the large N limit of $\mathcal{N} = 4$ SYM theory is equivalent to some classical theory. Maldacena has conjectured that this classical theory is IIB string theory on $AdS_5 \times S^5$, and that $1/N^2$ is equal to $\hbar_{\text{string theory}}$. There is now growing evidence for Maldacena's conjecture.

2.2.4 Interacting theory

The expectation value for the interacting theory is given by

$$
\langle \dots \rangle_{\text{int}} = \mathcal{N} \int [dM] e^{-\frac{\omega}{2} \operatorname{Tr}(M^2) - g\ \operatorname{Tr}(M^4)} \dots
$$

The following steps allow us to compute expectation values of traces in the interacting theory:

1. Couple in a source, $e^{\operatorname{Tr}(JM)}$:

$$
Z[J] = \mathcal{N} \int [dM] e^{-\frac{\omega}{2} \operatorname{Tr}(M^2) - g\ \operatorname{Tr}(M^4) + \operatorname{Tr}(JM)}
$$

   with

$$
\langle M_{ji} \rangle = \frac{d}{dJ_{ij}} Z[J]_{|J=0},
$$

   or more generally

$$
\langle M_{ij} M_{kl} \dots M_{mn} \rangle = \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} \dots \frac{d}{dJ_{nm}} Z[J]_{|J=0}.
$$

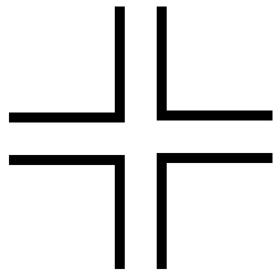
2. Expand the exponential, change M's into $\frac{d}{dJ}$'s and pull them out of the integral:

$$
Z[J] = \sum_{q=0}^{\infty} \frac{(-g)^q}{q!} \left(\frac{d}{dJ_{ba}} \frac{d}{dJ_{cb}} \frac{d}{dJ_{dc}} \frac{d}{dJ_{ad}} \right)^q \mathcal{N} \int [dM] e^{-\frac{\omega}{2} \operatorname{Tr}(M^2) + \operatorname{Tr}(JM)}.
$$

3. Perform the Gaussian integral by completing the square:

$$Z[J] = \sum_{q=0}^{\infty} \frac{(-g)^q}{q!} \left(\frac{d}{dJ_{ba}} \frac{d}{dJ_{cb}} \frac{d}{dJ_{dc}} \frac{d}{dJ_{ad}} \right)^q e^{\frac{1}{2\omega} \mathrm{Tr}\,(J^2)}.$$

Since this theory comes with a term that has "$-g\,\mathrm{Tr}\,(M^4)$", a new diagram is added with a new rule:



This is a vertex and since M is raised to the power of 4 in the trace, the vertex has 4 legs. Every vertex comes with a "$-g$".

2.2.5 Double scaling limit

Taking the limits $g \to 0$ and $N \to \infty$ such that $\lambda = gN$ is fixed and small, we see that

$$\langle \mathrm{Tr}\,(M^2) \rangle = \#_1 N^2 + \#_2 + \frac{\#_3}{N^2} + \frac{\#_4}{N^4} + \dots = N^2 \left(\#_1 + \frac{\#_2}{N^2} + \frac{\#_3}{N^4} + \dots \right),$$

where $\#_1 = C_0 + C_1\lambda + C_2\lambda^2 + C_3\lambda^3 + \dots$. Also,

$$\langle \mathrm{Tr}\,(M^4) \rangle = \#_1 N^3 + \#_2 N + \frac{\#_3}{N} + \dots = N^3 \left(\#_1 + \frac{\#_2}{N^2} + \frac{\#_3}{N^4} + \dots \right).$$

This expansion is called the *'t Hooft expansion* [6].

Note that $\hbar$ measures how difficult it is to simultaneously measure x and p. The variable "λ" is related to the string tension in string theory. It is a source of uncertainty and it is a new fundamental constant. There are two small numbers, $\frac{1}{N^2}$ (which is $\hbar$ for string theory) and λ (which is $\hbar$ for QFT). These numbers are a hint that you are doing string theory since in string theory we are faced with a new uncertainty principle which is connected with the string's tension, thus a new $\hbar$ is needed for the new uncertainty principle. The new uncertainty arises because the string is an extended object and hence is not able to probe position with the same resolution that a point particle can.

Note that factorisation does indeed hold true for the interacting theory, for example:

$$\langle \mathrm{Tr}(M^2)\,\mathrm{Tr}(M^2) \rangle_{\mathrm{int}} = \langle \mathrm{Tr}(M^2) \rangle_{\mathrm{int}} \langle \mathrm{Tr}(M^2) \rangle_{\mathrm{int}} \left(1 + O\left(\frac{1}{N^2} \right) \right).$$

As concluded before, from factorisation, a classical limit is achieved in the large N limit. Noting that $\frac{1}{N^2} \to 0$ since $N \to \infty$, we identify $\frac{1}{N^2} \equiv \hbar_{\text{large } N \text{ theory}} \to 0$. It will be argued that $\hbar$ of the new theory is $\hbar$ of string theory.

2.2.6 Rescaled variables

Recall that

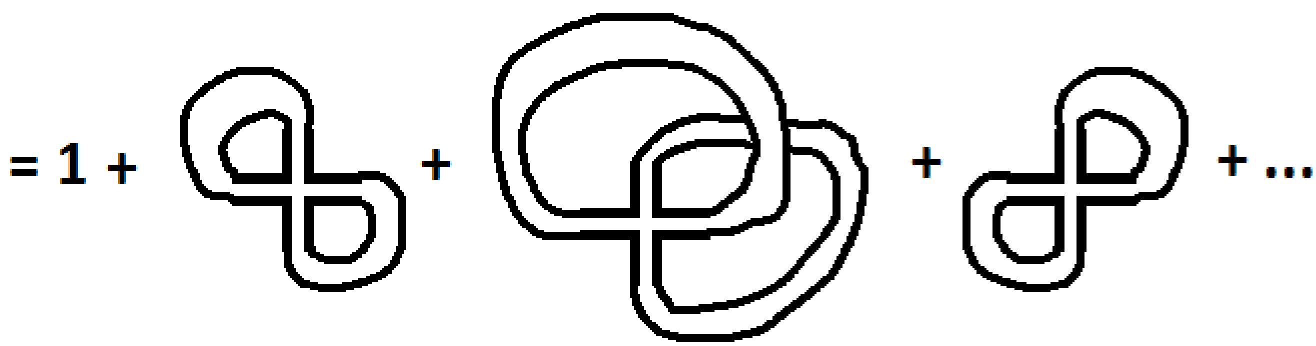
$$\langle \dots \rangle_{\text{int}} = \int [dM] e^{-\frac{\omega}{2}\,\mathrm{Tr}\,(M^2) - g\ \ \mathrm{Tr}\,(M^4)} \dots .$$

Now, let $M = \sqrt{N} M'$, so that $\frac{\omega}{2}\,\mathrm{Tr}\,(M^2) = \frac{N\omega}{2}\,\mathrm{Tr}\,(M'^2)$ and $g\,\mathrm{Tr}\,(M^4) = gN^2\,\mathrm{Tr}\,(M'^4) = N\lambda\,\mathrm{Tr}\,(M'^4)$. Thus,

$$\begin{aligned} Z[0] = \int [dM] e^{-\frac{\omega}{2}\,\mathrm{Tr}\,(M^2) - g\,\mathrm{Tr}\,(M^4)} &= \int [dM'] e^{-\frac{N\omega}{2}\,\mathrm{Tr}\,(M'^2) - \lambda N\,\mathrm{Tr}\,(M'^4)} \\ &= \int [dM'] e^{-\frac{N\omega}{2}\,\mathrm{Tr}\,(M'^2)} \sum_{n=0}^{\infty} (-\lambda N\,\mathrm{Tr}\,(M'^4))^n \frac{1}{n!}. \end{aligned}$$

Before, we had $\langle M_{ij} M_{kl} \rangle = \frac{\delta_{il}\delta_{jk}}{\omega}$ and for every M vertex we have a "$-g$". Now, we have $\langle M'_{ij} M'_{kl} \rangle = \frac{\delta_{il}\delta_{jk}}{N\omega}$ and for every M' vertex we have a "$-\lambda N$". Consider

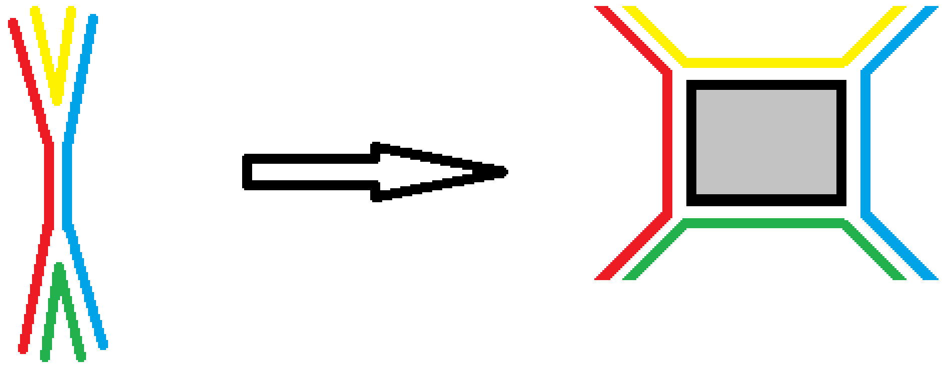
$$Z[0] = \int [dM'] e^{-\frac{N\omega}{2}\,\mathrm{Tr}\,(M'^2)} \left(1 + (-\lambda N)\,\mathrm{Tr}\,(M'^4) + \frac{(-\lambda N)^2}{2} (\mathrm{Tr}\,(M'^4))^2 + \dots \right) =$$



Looking at the first diagram, the value of it is $\left(\frac{1}{N\omega}\right)^2 (-\lambda N)(N)^3$. Now, the diagram has 2 edges, 1 vertex, and 3 faces, which we observe corresponds to the powers of 2, 1, and 3 that are of the terms $\left(\frac{1}{N\omega}\right)^2$, $(-\lambda N)^1$, and $(N)^3$ respectively. In general, the Feynman diagrams will have the value

$$\left(\frac{1}{N\omega}\right)^E (-\lambda N)^V (N)^F,$$

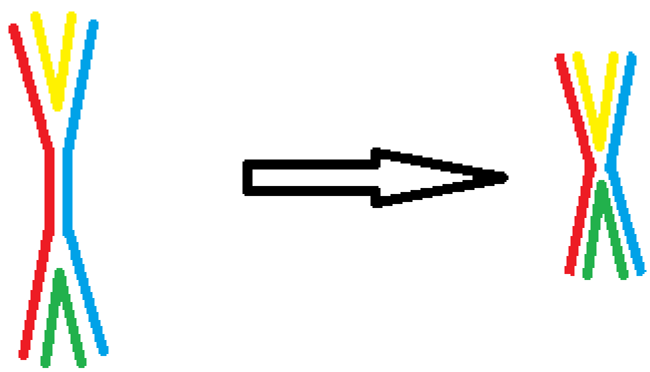
where E, V, and F correspond to the number of edges, vertices, and faces a diagram has respectively. The N dependence of a graph with E edges, F faces, and V vertices is N^{F-E+V}. It should be noted that $\chi \equiv F - E + V$ is called the *Euler characteristic* and it is a topological invariant which means that all spaces of the same topology share the same number. For example, modify a triangulation by stretching it such that we create a new face:



After the stretch, we get $F' = F + 1$, $E' = E + 3$, and $V' = V + 2$, and then

$$F' - E' + V' = F + 1 - (E + 3) + V + 2 = F - E + V,$$

which is a concrete demonstration that this quantity is a topological invariant. If we squish the triangulation to get rid of an edge, we find

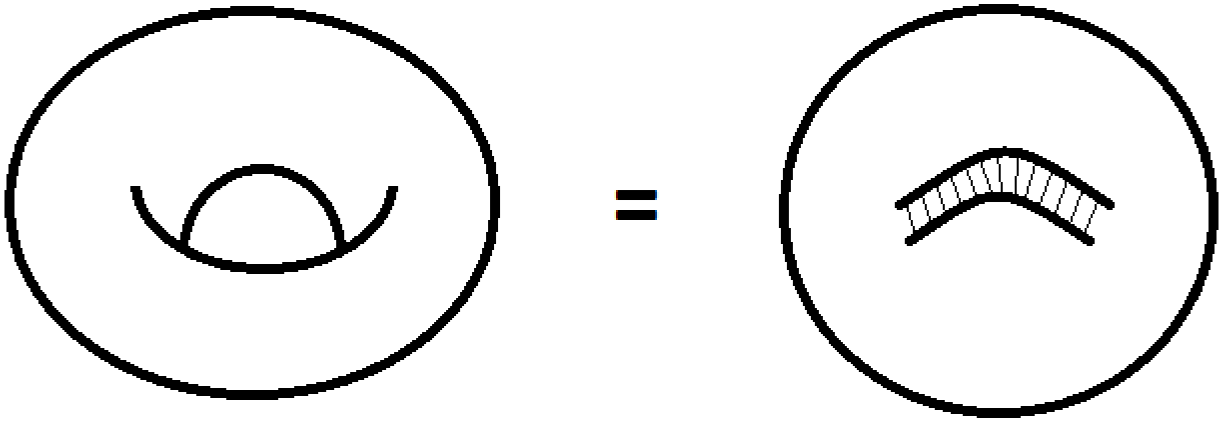


After the squish, we get $F'' = F$, $E'' = E - 1$, and $V'' = V - 1$, and then

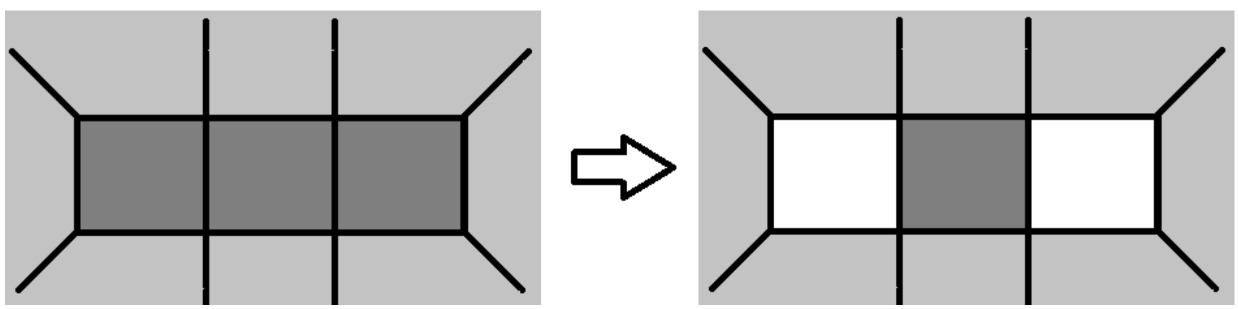
$$F' - E' + V' = F - (E - 1) + (V - 1) = F - E + V,$$

which is also a topological invariant.

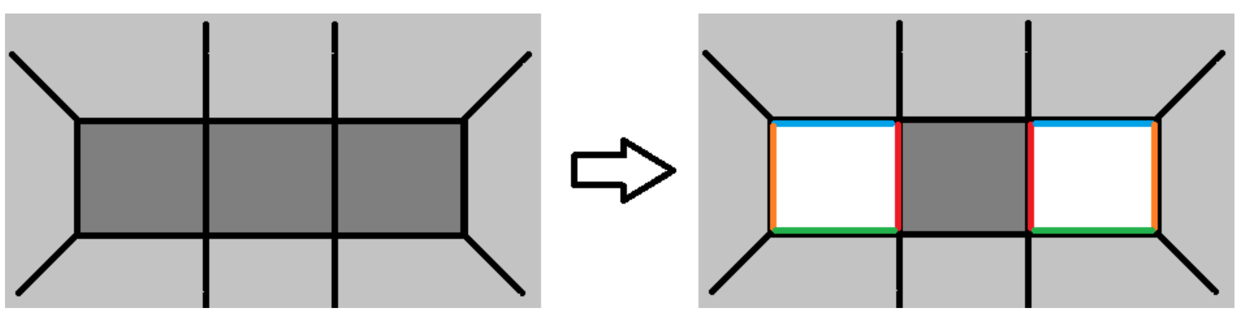
A torus is obtained by modifying a sphere by adding a handle:



For us to add a handle to a sphere, we need to cut 2 holes:



This gives us $F' = F - 2$, $E' = E$, and $V' = V$. Next, add the handle:

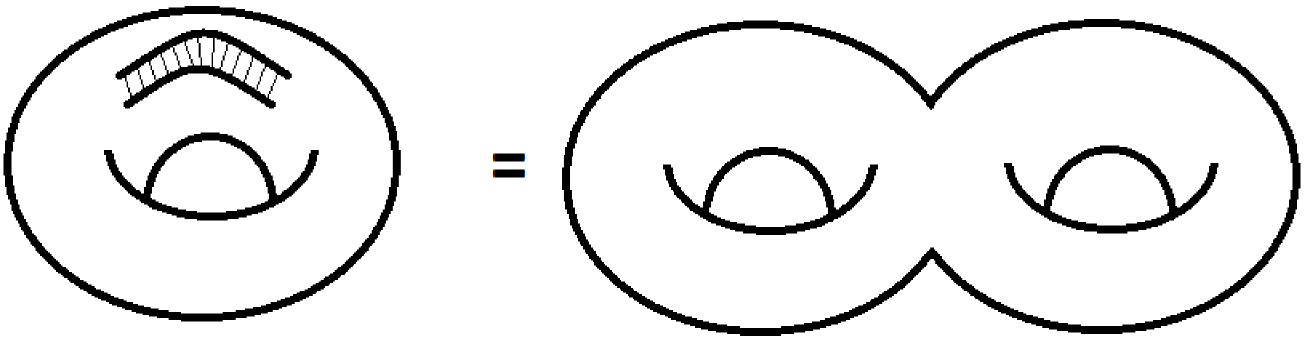


This gives us $F'' = F - 2$, $E'' = E - 4$, and $V'' = V - 4$. Thus,

$$F - E + V \to F'' - E'' + V'' = F - E + V - 2.$$

This tells you that you will always subtract 2 from χ each time you add a handle, just as you do to create a torus from a sphere.

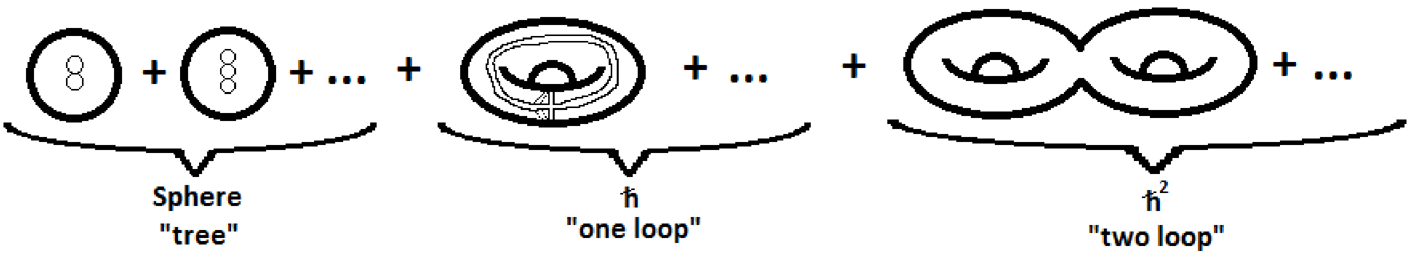
Adding a handle to a torus produces a pretzel:



Note the following:

- $\chi_{\text{sphere}} = F_{\text{sphere}} - E_{\text{sphere}} + V_{\text{sphere}} = 4 - 6 + 4 = 2$
- $\chi_{\text{torus}} = \chi_{\text{sphere}} - 2 = 0$
- $\chi_{\text{pretzel}} = \chi_{\text{torus}} - 2 = -2$

The Feynman diagrams we get from the interacting theory triangulate the surface (that being a sphere, torus, pretzel, etc.). Thus, the following are surfaces that are triangulated by diagrams produced from computing correlators.



Our path integral can be reinterpreted as a sum over surfaces. For particles we sum over world lines. For strings we sum over world sheets which is a sum over surfaces.

2.2.7 Complex matrices

Suppose we construct a complex matrix Z using two $N \times N$ Hermitian matrices M_1 and M_2 such that $Z = \frac{1}{\sqrt{2}}(M_1 + iM_2)$ and $Z^\dagger = \frac{1}{\sqrt{2}}(M_1 - iM_2)$. The correlators for this matrix model are defined by

$$\langle \dots \rangle = \int [dZ\ dZ^\dagger] e^{-\omega \operatorname{Tr}(ZZ^\dagger)} \dots = \int [dM_1\ dM_2] e^{-\frac{\omega}{2}\operatorname{Tr}(M_1^2) - \frac{\omega}{2}\operatorname{Tr}(M_2^2)} \dots\ .$$

We can evaluate the following 3 important correlators:

- $\langle Z_{ij} Z_{kl} \rangle = 0$

- $\langle Z^\dagger_{ij} Z^\dagger_{kl} \rangle = 0$
- $\langle Z_{ij} Z^\dagger_{kl} \rangle = \frac{1}{\omega} \delta_{il} \delta_{jk}$.

Now, computing correlators of traces of Z, where we group fields in pairs (as in round brackets in the following equation), we see that

$$\begin{aligned}
\langle \operatorname{Tr}(Z^2) \operatorname{Tr}(Z^{\dagger^2}) \rangle &= \langle (Z_{ij} Z_{ji})(Z^\dagger_{kl} Z^\dagger_{lk}) \rangle + \langle (Z_{ij} Z^\dagger_{kl})(Z_{ji} Z^\dagger_{lk}) \rangle + \langle (Z_{ji} Z^\dagger_{kl})(Z_{ij} Z^\dagger_{lk}) \rangle \\
&= \langle (Z_{ij} Z^\dagger_{kl})(Z_{ji} Z^\dagger_{lk}) \rangle + \langle (Z_{ji} Z^\dagger_{kl})(Z_{ij} Z^\dagger_{lk}) \rangle
\end{aligned}$$

which produces 2 diagrams. Similarly, $\langle \operatorname{Tr}(Z^3) \operatorname{Tr}(Z^{\dagger^3}) \rangle$ will produce 6 graphs. It can be deduced that $\langle \operatorname{Tr}(Z^n) \operatorname{Tr}(Z^{\dagger^n}) \rangle$ will produce $n!$ diagrams. The first two correlators are illustrated with diagrams below:

which are equal to $\frac{2N^2}{\omega^2}$ and $\frac{3N^3}{\omega^3} + \frac{3N}{\omega^3}$ respectively. In the large N limit:

$$\begin{aligned}
\langle \operatorname{Tr}(Z^2) \operatorname{Tr}(Z^{\dagger^2}) \rangle &= \frac{2N^2}{\omega^2} \\
\langle \operatorname{Tr}(Z^3) \operatorname{Tr}(Z^{\dagger^3}) \rangle &= \frac{3N^3}{\omega^3} \left(1 + O\left(\frac{1}{N^2} \right) \right).
\end{aligned}$$

The general form of the correlator for traces of complex matrices is

$$\langle \operatorname{Tr}(Z^J) \operatorname{Tr}(Z^{\dagger^J}) \rangle = \frac{J N^J}{\omega^J} \left(1 + O\left(\frac{1}{N^2} \right) \right).$$

Planar diagrams are diagrams you can draw on a plane/sphere. Only the planar diagrams are counted in the large N limit since only $N^{\chi_{\text{sphere}}} = N^2$ dominates in the large N limit. Introducing an operator $\mathcal{O}_{\mathcal{J}}$ such that

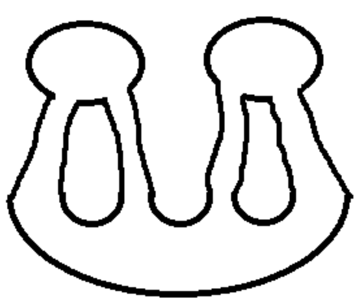
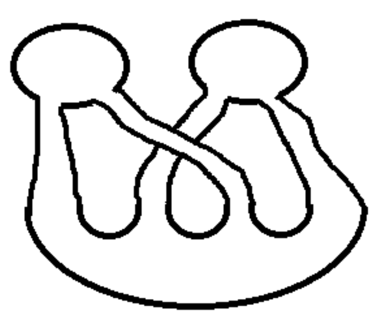
$$\mathcal{O}_{\mathcal{J}} = \frac{\sqrt{\omega^J} \operatorname{Tr}(Z^J)}{\sqrt{J N^J}},$$

we find the correlator of this operator is

$$\langle \mathcal{O}_J \mathcal{O}_K^\dagger \rangle = \delta_{JK}.$$

We now wish to evaluate a correlator of the form $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_4^\dagger \rangle$:

$$\begin{aligned}
\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_4^\dagger \rangle &= \langle \frac{\omega}{\sqrt{2}N} \operatorname{Tr}(Z^2) \frac{\omega}{\sqrt{2}N} \operatorname{Tr}(Z^2) \frac{\omega^2}{2N^2} \operatorname{Tr}(Z^{\dagger^4}) \rangle \\
&= \frac{\omega^4}{4N^4} \langle \operatorname{Tr}(Z^2) \operatorname{Tr}(Z^2) \operatorname{Tr}(Z^{\dagger^4}) \rangle
\end{aligned}$$

$\equiv \mathbf{A} \Big[$ **16**  **+ 8**  $\Big]$

$$\begin{aligned}
&= A \left[\frac{16N^3}{\omega^3} + \text{non-planar} \right] \\
&= A \left[\frac{2 \times 2 \times 4N^3}{\omega^3} + \text{non-planar} \right]
\end{aligned}$$

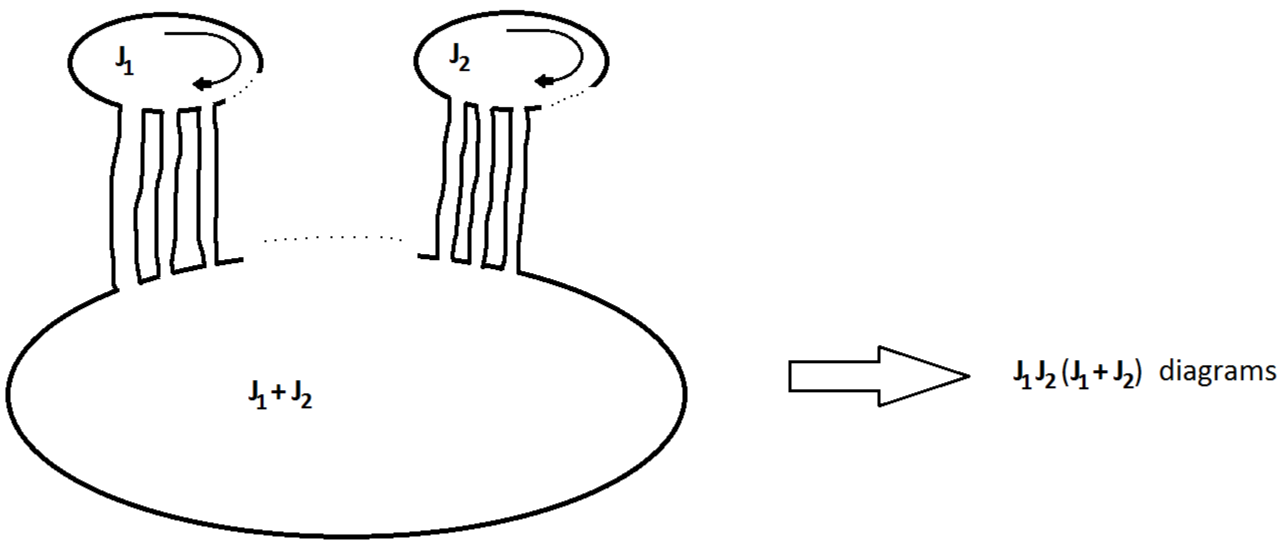
where $A \equiv \frac{\omega^4}{4N^4}$. The general form of the example above is

$$\langle \operatorname{Tr}(Z^{J_1}) \operatorname{Tr}(Z^{J_2}) \operatorname{Tr}(Z^{\dagger^{J_1+J_2}}) \rangle = \frac{J_1 J_2 (J_1 + J_2) N^{J_1+J_2-1}}{\omega^{J_1+J_2}}$$

and

$$\begin{aligned}
\langle \mathcal{O}_{J_1} \mathcal{O}_{J_2} \mathcal{O}_K^\dagger \rangle &= \delta_{K,J_1+J_2} \frac{\sqrt{\omega^{J_1+J_2+K}}}{\sqrt{J_1 J_2 K}\sqrt{N^{J_1+J_2+K}}} \langle \operatorname{Tr}(Z^{J_1}) \operatorname{Tr}(Z^{J_2}) \operatorname{Tr}(Z^{\dagger^K}) \rangle \\
&= \delta_{K,J_1+J_2} \frac{\omega^{J_1+J_2}}{\sqrt{J_1 J_2 (J_1+J_2)}} \frac{N^{J_1+J_2-1}}{N^{J_1+J_2}} \frac{J_1 J_2 (J_1+J_2)}{\omega^{J_1+J_2}} \\
&= \delta_{K,J_1+J_2} \frac{\sqrt{J_1 J_2 (J_1+J_2)}}{N},
\end{aligned}$$

since $f(x)\delta_{x,a} = f(a)\delta_{x,a}$. Note that we only sum the planar diagram contribution since it is the only contribution that survives in the large N limit. To illustrate this result, we have two small loops that have J_1 and J_2 legs respectively, and they connect to a bigger loop that has $J_1 + J_2$ legs:



The J_1 and J_2 diagrams can have their legs shifted around the J_1+J_2 diagram producing J_1 and J_2 diagrams respectively. But the J_1+J_2 diagram can also shift its legs resulting in J_1+J_2 diagrams. Therefore, in total, $J_1J_2(J_1+J_2)$ diagrams are produced. This diagram can be analogously thought of as 2 strings with lengths J_1, J_2 joining to produce a third string of length J_1+J_2.

2.2.8 Comparisons

In this section we will compare the results from the matrix model to those of $\mathcal{N}=4$ SYM theory. To go from the matrix model results to the $\mathcal{N}=4$ SYM results, we need to append spacetime dependence which is easily determined by dimensional analysis. This section will provide concrete examples which nicely illustrate this.

<u>Matrix Model:</u>

Recall that for the matrix model, the correlator is

$$\langle Z_{ij}Z^\dagger_{kl}\rangle = \frac{\delta_{il}\delta_{jk}}{\omega}$$

and the operator we use is

$$\mathcal{O}_J = \frac{\sqrt{\omega^J}\,\mathrm{Tr}\,(Z^J)}{\sqrt{JN^J}},$$

with the correlators

$$\langle \mathcal{O}_J\mathcal{O}^\dagger_K\rangle = \delta_{JK}$$

and

$$\langle \mathcal{O}_{J_1}\mathcal{O}_{J_2}\mathcal{O}^\dagger_K\rangle = \delta_{K,J_1+J_2}\frac{\sqrt{J_1J_2(J_1+J_2)}}{N}.$$

$\mathcal{N} = 4$ SYM:

In this theory, the action is

$$S = \int d^4x \;\; \mathrm{Tr}\,(\partial_\mu Z \partial^\mu Z^\dagger) + \ldots$$

where $Z = \phi_1 + i\phi_2$, and ϕ_1 and ϕ_2 are hermitian matrices. Thus, the correlator in this theory is

$$\langle Z_{ij}(x_1) Z^\dagger_{kl}(x_2)\rangle = \frac{\delta_{il}\delta_{jk}}{|x_1 - x_2|^2}.$$

The gauge invariant operators for this theory include

$$\mathcal{O}_J(x_1) = \frac{\mathrm{Tr}\,(Z^J(x_1))}{\sqrt{JN^J}},$$

and thus, the correlators we study are

$$\langle \mathcal{O}_J(x_1)\mathcal{O}_K(x_2)\rangle = \frac{\delta_{JK}}{|x_1 - x_2|^{2J}},$$

and

$$\langle \mathcal{O}_{J_1}(x_1)\mathcal{O}_{J_2}(x_1)\mathcal{O}^\dagger_K(x_2)\rangle = \delta_{K,J_1+J_2}\frac{\sqrt{J_1 J_2 (J_1 + J_2)}}{N}\frac{1}{|x_1 - x_2|^{2(J_1+J_2)}}.$$

2.2.9 Link between string theory and gauge theory

It is claimed that for IIB strings on $AdS_5 \times S^5$, energy and momentum on S^5 is mapped into the dimension Δ/scaling dimension and R-charge in $\mathcal{N} = 4$ SYM respectively. The scaling dimension of any local operator A is equal to the negative power of the dimension of length at the free field fixed point. To be explicit, $\Delta_A = B \Rightarrow [A] = L^{-B}$.

Here are some more simple examples:

- $[Z] = L^{-1} \Rightarrow \Delta_Z = 1$
- $[Z^2] = L^{-2} \Rightarrow \Delta_{Z^2} = 2$

Thus the scaling dimension for the operator $\mathcal{O}_J$ is

$$\Delta_{\mathcal{O}_J} = J,$$

since

$$[\mathcal{O}_J] = [\mathrm{Tr}\,(Z^J)] = [Z^J] = L^{-J}.$$

The scaling dimension of operator $\mathcal{O}_J$ is in fact the energy of the dual string state as stated before, i.e. energy in string theory $\leftrightarrow \Delta$ in $\mathcal{N} = 4$ SYM.

A 5-sphere, S^5, defined by $(x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 + (x_5)^2 + (x_6)^2 = R^2$ has an $SO(6)$

symmetry. For the QFT, there are the 6 fields $\phi_1, \ldots, \phi_6$ that also enjoy an $SO(6)$ symmetry which is called the R-symmetry. Remember that the angular momentum operator is $L = -i\frac{\partial}{\partial\phi}$ and $e^{i\theta L}Z = e^{i\theta\#}Z$, where $Z = \phi_1 + i\phi_2$ and $\#$ is equal to the charge Z carries, also called the R-charge. We rotate Z such that

$$\phi_1 \to \phi_1' = \cos(\theta)\phi_1 + \sin(\theta)\phi_2$$

and

$$\phi_2 \to \phi_2' = -\sin(\theta)\phi_1 + \cos(\theta)\phi_2,$$

thus

$$\begin{aligned}
\phi_1' + i\phi_2' &= \cos(\theta)\phi_1 + \sin(\theta)\phi_2 - i\sin(\theta)\phi_1 + i\cos(\theta)\phi_2 \\
&= e^{-i\theta}\phi_1 + ie^{-i\theta}\phi_2 \\
&= e^{-i\theta}(\phi_1 + i\phi_2) \\
&= e^{-i\theta}Z.
\end{aligned}$$

Therefore, we deduce that $\#$ for Z is -1. We have thus also deduced that the R-charge of $\mathcal{O}_J$ is

$$R_{\mathcal{O}_J} = -J,$$

which is in fact the momentum of the string as stated before, i.e. momentum in string theory $\leftrightarrow$ R-charge in $\mathcal{N} = 4$ SYM. Now, we can deduce the mass of $\mathcal{O}_J$:

$$\begin{aligned}
m^2_{\mathcal{O}_J} &= E^2_{\mathcal{O}_J} - p^2_{\mathcal{O}_J} \\
&= \Delta^2_{\mathcal{O}_J} - R^2_{\mathcal{O}_J} \\
&= J^2 - (-J)^2 \\
&= 0.
\end{aligned}$$

Therefore, $\mathcal{O}_J$ corresponds to a massless graviton moving on a 5-sphere.

2.2.10 Fock space in supergravity

Remember that the field constructed in QFT with relativistic normalisation is given as

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}(e^{-ik\cdot x}\alpha(k) + e^{ik\cdot x}\alpha^\dagger(k)),$$

where $\alpha(k) = (2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{k}}}a_{\vec{k}}$ and $\alpha^\dagger(k) = (2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{k}}}a^\dagger_{\vec{k}}$. For the non-relativistic case, we had $\langle\vec{k}'|\vec{k}\rangle = \delta(\vec{k}-\vec{k}')$, $a^\dagger_{\vec{k}}|0\rangle = |\vec{k}\rangle$, and $a^\dagger_{\vec{k_1}}a^\dagger_{\vec{k_2}}|0\rangle = |\vec{k_1}, \vec{k_2}\rangle$ for the continuous version, and $\langle n|m\rangle = \delta_{nm}$, $a^\dagger_n|0\rangle = |n\rangle$, and $a^\dagger_n a^\dagger_m|0\rangle = |n, m\rangle$ for the discrete version.

Now, $|J\rangle$ = state of graviton with energy J and momentum $-J$. We also have $\langle J|K\rangle = \delta_{JK}$ and $\langle J_1, J_2|J_1 + J_2\rangle = 0$ (since a one particle state is orthogonal to a two particle state in the free theory). When energies become very high, then there is interaction between the particle states. This is due to gravity, and the "charge" responsible for the generating of the gravitational field is the mass. We know how mass and energies are related due to Einstein's formula $E = mc^2$. In this case we will start to see that $\langle J_1, J_2|J_1+J_2\rangle$ begins to differ from zero. Computing $\langle J_1, J_2|J_1+J_2\rangle$, we have

$$\langle J_1, J_2|J_1+J_2\rangle = \delta_{K,J_1+J_2}\frac{\sqrt{J_1J_2(J_1+J_2)}}{N},$$

which is obtained by summing only the planar contribution (meaning it is not exact) and it suggests operators are orthogonal as long as $J \ll N^{\frac{2}{3}}$. Considering the case when $J_1, J_2 = O(1)$, the correlator is

$$\langle J_1, J_2|J_1+J_2\rangle = \delta_{K,J_1+J_2}\frac{\sqrt{J_1J_2(J_1+J_2)}}{N} \to \delta_{K,J_1+J_2}\frac{\#}{N} \to 0, \text{ as } N\to\infty.$$

However, when $J_1, J_2 = O(N)$, then the correlator reads

$$\langle J_1, J_2|J_1+J_2\rangle = \delta_{K,J_1+J_2}\frac{\sqrt{J_1J_2(J_1+J_2)}}{N} \to \delta_{K,J_1+J_2}\frac{\sqrt{N^2}}{N}\left(\#\sqrt{N}\right) = \#\sqrt{N}\delta_{K,J_1+J_2} \neq 0.$$

Thus, the identification of each trace with a particle has broken down. In the matrix model, we have

$$\langle \mathcal{O}_J\mathcal{O}_J^\dagger\rangle = JN^J + J^5N^{J-2} + \dots$$

Note that if $J \sim N^{\frac{1}{2}}$, then the N^{J-2} term is as big as the N^J term. As a result of that, you then cannot neglect the non-planar diagrams; you *must* sum everything. This informs us that as you change J, the object you are studying changes. To make clear what was just stated, the table below illustrates the relationship between J (which is a label of QFT) and the object (which lives in string theory):

J	object
$O(1)$	graviton
$O(\sqrt{N})$	string
$O(N)$	giant graviton
$O(N^2)$	new spacetime geometry

To explain this further in detail, the table below explains what each order means in connection to J and the dual string theory object:

N	graviton ($O(1)$)	string ($O(\sqrt{N})$)	giant graviton ($O(N)$)	new $g_{\mu\nu}$ ($O(N^2)$)
10	$\mathrm{Tr}\,(Z^3)$	$\mathrm{Tr}\,(Z^3Y)$	$\mathrm{Tr}\,(Z^4)+\dots$	$\mathrm{Tr}\,(Z^4)+\dots$
100	$\mathrm{Tr}\,(Z^3)$	$\mathrm{Tr}\,(Z^9Y)$	$\mathrm{Tr}\,(Z^{40})+\dots$	$\mathrm{Tr}\,(Z^{400})+\dots$
1000	$\mathrm{Tr}\,(Z^3)$	$\mathrm{Tr}\,(Z^{27}Y)$	$\mathrm{Tr}\,(Z^{400})+\dots$	$\mathrm{Tr}\,(Z^{40\ 000})+\dots$

For a graviton, we use planar diagrams only. For a string we use planar diagrams alone as long as $\frac{J^2}{N} \ll 1$. For a giant graviton, every diagram contributes; there will be $N!$ diagrams and we take $N\to\infty$. For new spacetime geometry, if it is understood, then one will understand everything about black holes; in this case too, we need to sum all of the diagrams.

2.2.11 Conclusion

When constructing the matrix model, we construct Gaussian integrals that help us to evaluate correlators. This led us to specific properties of correlators which compelled us to think of making diagrams to help when computing correlators; we called these diagrams *ribbon diagrams*. These diagrams came with their own set of Feynman rules and we deduced that traces of powers of our matrices are the physical observables with interesting correlators.

When we take the limit $N \to \infty$, we notice that only the leading term of the correlators survive and that the expectation value of products is equal to the product of expectation values. This result was first obtained by [7] and he called this concept *factorisation*. We also linked this to the conjecture that the large N limit of $\mathcal{N} = 4$ SYM theory, is given by the classical limit of IIB string theory on $AdS_5 \times S^5$, and we argued that $1/N^2$ is equal to $\hbar_{\text{string theory}}$.

We also considered Gaussian integrals for the interacting theory. The interaction term "$-g\ \mathrm{Tr}\,(M^4)$" required a new diagram: the vertex ribbon diagram that has 4 legs corresponding to M to the power of 4. From there we take the double scaling limit $g \to 0$ and $N \to \infty$ such that $\lambda = gN$ is fixed and small, and this lead us to the result known as the *'t Hooft expansion*. The new constant λ along with $1/N^2$ are both related to $\hbar$ (QFT and string theory respectively) and they indicate that studying the large N expansion in the matrix model is equivalent to doing string theory.

We rescaled variables such that ribbon graphs had a N dependence of N^{F-E+V} where $\chi = F - E + V$ is called the *Euler characteristic*. It is a topological invariant. We came to the conclusion that the ribbon diagrams of the interacting theory triangulate a surface.

Following that, we investigated complex matrices and the correlators associated with complex matrices. We noted that $\mathrm{Tr}\,(Z^n)\,\mathrm{Tr}\,(Z^{\dagger^n})$ will produce $n!$ diagrams, but in the large N limit, we will only get contributions from planar diagrams. Thus $\langle \mathrm{Tr}\,(Z^J)\,\mathrm{Tr}\,(Z^{\dagger^J})\rangle = \frac{JN^J}{\omega^J}\left(1 + O\left(\frac{1}{N^2}\right)\right)$ as $N \to \infty$. We introduced the operator $\mathcal{O}_J$ which normalises $\mathrm{Tr}\,(Z^J)$ such that $\langle \mathcal{O}_J \mathcal{O}_K\rangle = \delta_{JK}$. Through further investigation, we also found that $\langle \mathcal{O}_{J_1}\mathcal{O}_{J_2}\mathcal{O}^\dagger_K\rangle = \delta_{K,J_1+J_2}\frac{\sqrt{J_1J_2(J_1+J_2)}}{N}$, where J_1, J_2, and $J_1 + J_2$ can be thought of as lengths of 3 strings. Comparing this result from the matrix model to $\mathcal{N} = 4$ SYM, we see that in $\mathcal{N} = 4$ SYM theory, $\langle \mathcal{O}_{J_1}(x_1)\mathcal{O}_{J_2}(x_1)\mathcal{O}^\dagger_K(x_2)\rangle = \delta_{K,J_1+J_2}\frac{\sqrt{J_1J_2(J_1+J_2)}}{N}\frac{1}{|x_1-x_2|^{2(J_1+J_2)}}$. It has the additional term of $\frac{1}{|x_1-x_2|^{2(J_1+J_2)}}$ compared to the result of the matrix model correlator. From there, we deduced that the scaling dimension of $\mathcal{O}_J$ is $\Delta_{\mathcal{O}_J} = J$ and the R-charge of $\mathcal{O}_J$ is $R_{\mathcal{O}_J} = -J$, which relates to the energy and momentum in the string theory respectively. Thus we determine the mass of $\mathcal{O}_J$ to be $m^2_{\mathcal{O}_J} = E^2_{\mathcal{O}_J} - p^2_{\mathcal{O}_J} = 0$, which shows that $\mathcal{O}_J$ is a massless graviton moving on a 5-sphere.

In the Fock space in supergravity, when the energy is very high, there will be interactions between particle states. The correlator we study becomes $\langle \mathcal{O}_J\mathcal{O}^\dagger_J\rangle = JN^J + J^5N^{J-2} + \dots$, and if $J \sim N^{\frac{1}{2}}$, then the N^{J-2} term is as big as the N^J term, so one cannot neglect the non-planar diagrams. It further shows us that changing J changes the interpretation of the object you study (in string theory). Taken together, this is compelling evidence that matrix models are in fact string theories.

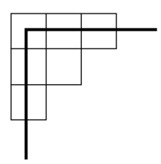
2.3 Young diagrams

We are trying to go beyond the planar limit since it is required when studying an emergent Yang-Mills theory. With the use of group representation theory, we will be able to sum all ribbon graphs and not just planar contributions. Young diagrams play a central role in this discussion.

A Young diagram is an array of n boxes aligned in rows starting from the left to right and each consecutive row will have the same or less boxes as the row above it. Young diagrams label all of the possible irreducible inequivalent representations (otherwise known as irreps) of the symmetric group S_n. When we list the number of boxes in each row of Young diagram R, it gives a partition of n, so we can write "R is a partition of n", which is denoted as $R \vdash n$. Each representation is a set of matrices acting on a vector space. We can label a basis for this vector space using the Young-Yamanouchi symbols. For us to get a Young-Yamanouchi symbol, we fill in the boxes with the integers $1, 2, 3, \ldots, n$ in the order of which box we could remove such that what is left is still a valid Young diagram. For example, suppose we have irrep labelled by $\yng(3,1)$ which has the following Young-Yamanouchi symbols:

$$\young(432,1) \quad \young(421,3) \quad \young(431,2)\ .$$

There is a formula to determine the dimensions of a symmetric group irrep given the Young diagram. This formula includes the notion of a hook length, which is a number that is given to each box in the Young diagram. The hook length of a box b in Young diagram R is the number of boxes that are in the same row to the right of b plus the number of boxes in the same column below b plus one, which is for including b itself. Inside the column containing box b, draw a line from below the bottom of R to b and then continue that line to the right until you exit R. The hook length is equal to the number of boxes this line (the "hook") passes through. The diagram below illustrates an example of a hook associated to the first box in the first row of the Young diagram $\yng(3,2,1)$ that has a hook length of 5:



As an example, the Young diagram $\yng(3,2,1)$ with the hooks lengths filled in is given as

$$\text{hook lengths} = \young(531,31,1)\ .$$

Now, the dimension of an irrep labelled by Young diagram R is equal to $n!$ divided by the product of hook lengths, and it is written as

$$d_R = \frac{n!}{\prod_{x \in R} \text{hook}(x)} \equiv \frac{n!}{\text{hooks}_R},$$

where $R \vdash n$. The Young diagram $\yng(3,2,1)$ labels an irrep of S_6 with dimension

$$d_{\yng(3,2,1)} = \frac{6!}{5 \cdot 3 \cdot 1 \cdot 3 \cdot 1 \cdot 1} = 16.$$

Thus, there are 16 valid Young-Yamanouchi symbols that can be drawn. We now introduce the concept of the *content* of a box in a Young diagram. A box x in row i and column j of R has content $j - i$. Here is an example of Young diagram with the content filled in

$$\begin{ytableau} 0 & 1 & 2 \\ -1 & 0 \\ -2 \end{ytableau}$$

For a Young-Yamanouchi symbol that is given, each box in the Young diagram is labelled by an integer i that is unique, with $1 \leq i \leq n$. The content of the box labelled i is c_i. Let $|R_{(k,k+1)}\rangle$ denote the Young-Yamanouchi symbol that is obtained from $|R\rangle$ by swapping the labels of boxes k and $k+1$. The matrix elements of the adjacent transpositions are now specified by

$$\Gamma_R((k,k+1))|R\rangle = \frac{1}{c_k - c_{k+1}}|R\rangle + \sqrt{1 - \frac{1}{(c_k - c_{k+1})^2}}|R_{(k,k+1)}\rangle.$$

An example of using this formula is given below:

$$\Gamma_{\ydiagram{2,1,1}}((1\ 2))\left|\begin{ytableau} 4 & 3 \\ 2 \\ 1 \end{ytableau}\right\rangle = -\left|\begin{ytableau} 4 & 3 \\ 2 \\ 1 \end{ytableau}\right\rangle$$

$$\Gamma_{\ydiagram{2,1,1}}((1\ 2))\left|\begin{ytableau} 4 & 2 \\ 3 \\ 1 \end{ytableau}\right\rangle = -\frac{1}{3}\left|\begin{ytableau} 4 & 2 \\ 3 \\ 1 \end{ytableau}\right\rangle + \frac{\sqrt{8}}{3}\left|\begin{ytableau} 4 & 1 \\ 3 \\ 2 \end{ytableau}\right\rangle$$

$$\Gamma_{\ydiagram{2,1,1}}((1\ 2))\left|\begin{ytableau} 4 & 1 \\ 3 \\ 2 \end{ytableau}\right\rangle = \frac{1}{3}\left|\begin{ytableau} 4 & 1 \\ 3 \\ 2 \end{ytableau}\right\rangle + \frac{\sqrt{8}}{3}\left|\begin{ytableau} 4 & 2 \\ 3 \\ 1 \end{ytableau}\right\rangle.$$

By choosing

$$\left|\begin{ytableau} 4 & 3 \\ 2 \\ 1 \end{ytableau}\right\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \left|\begin{ytableau} 4 & 2 \\ 3 \\ 1 \end{ytableau}\right\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \left|\begin{ytableau} 4 & 1 \\ 3 \\ 2 \end{ytableau}\right\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

we find the following matrix representation for the permutation $(1\ 2)$

$$\Gamma_{\ydiagram{2,1,1}}((1\ 2)) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{\sqrt{8}}{3} \\ 0 & \frac{\sqrt{8}}{3} & \frac{1}{3} \end{bmatrix}.$$

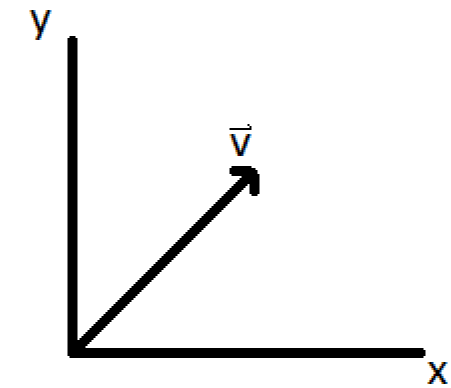
So in conclusion, we learn that Young diagrams label irreps, they provide us with a formula for the dimensions of irreps, they can be used to label a basis and they give us matrices of irreps.

2.4 Single Matrix Z

As will become clear below, in applying group representation theory to matrix models we will make frequent use of projection operators. In this section we will introduce some of the projection operators that are needed.

2.4.1 Projection operators

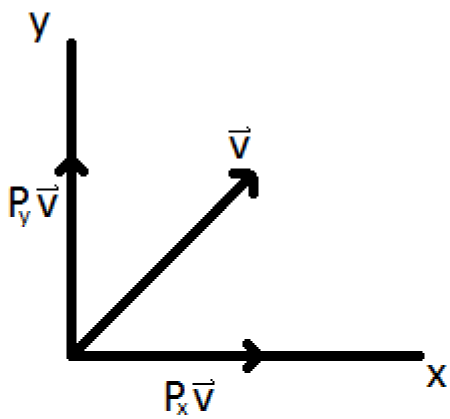
Suppose we have a vector $\vec{v}$ that has two components $\vec{v} = (v_x, v_y)$. The diagram below illustrates this arbitrary vector



Introduce the operators $\hat{P}_x$ and $\hat{P}_y$ such that

$$\hat{P}_x \vec{v} = (v_x, 0) \quad , \quad \hat{P}_y \vec{v} = (0, v_y).$$

The diagram below makes this more clear:



$\hat{P}_x$ and $\hat{P}_y$ are projection operators that project any arbitrary vector $\vec{v}$ to its x-component and y-component respectively in the plane. It is important to note the following: $\hat{P}_x \cdot \hat{P}_x |v\rangle = \hat{P}_x^2 |v\rangle = \hat{P}_x |v\rangle$ which implies

$$\hat{P}_x^2 = \hat{P}_x,$$

and similarly $\hat{P}_y^2 = \hat{P}_y$. Also, one should note that

$$\hat{P}_x \cdot \hat{P}_y = \hat{P}_y \cdot \hat{P}_x = 0.$$

This result is simple to interpret: $\hat{P}_x \cdot \hat{P}_y$ says you first project onto the y-axis and from there you project onto the x-component of the y-axis which is the origin. A similar argument is made for $\hat{P}_y \cdot \hat{P}_x$. The final result to note is that

$$\hat{P}_x |v\rangle + \hat{P}_y |v\rangle = |v\rangle$$
$$\Rightarrow \hat{P}_x + \hat{P}_y = \mathbb{1}.$$

This is obvious upon noting that $\hat{P}_x\vec{v} = (v_x, 0)$ and $\hat{P}_y\vec{v} = (0, v_y)$. These properties of projection operators, that are obvious in our simple example, hold in general. We can define projection operators that project onto definite irreducible representations of the symmetric group. The projector that projects onto irrep R is

$$(\hat{P}_R)^I_J = \frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)(\sigma)^I_J,$$

where d_R is the dimension of irrep R, σ is a permutation, S_n is the symmetric group permuting n elements, $\chi_R(\sigma)$ is the character of σ in irrep R, I and J denote $i_1i_2\ldots i_n$ and $j_1i_2\ldots j_n$ respectively, and $(\sigma)^I_J = \delta^{i_1}_{j_{\sigma(1)}}\ldots\delta^{i_n}_{j_{\sigma(n)}}$.

We will now consider the cases when $n = 2$ with $R = \Box\Box$ and $R = \genfrac{}{}{0pt}{}{\Box}{\Box}$:

- $n = 2$ (thus $S_2 = \{\mathbb{1}, (1\ 2)\}$) and $R = \Box\Box$

$$\begin{aligned}
(\hat{P}_{\Box\Box})^{i_1i_2}_{j_1j_2}v^{j_1}w^{j_2} &= \frac{1}{2}\left[\chi_{\Box\Box}(\mathbb{1})(\mathbb{1})^{i_1i_2}_{j_1j_2} + \chi_{\Box\Box}((1\ 2))((1\ 2))^{i_1i_2}_{j_1j_2}\right]v^{j_1}w^{j_2}\\
&= \frac{1}{2}\left[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\right]v^{j_1}w^{j_2}\\
&= \frac{1}{2}\left[v^{i_1}w^{i_2} + w^{i_1}v^{i_2}\right],
\end{aligned}$$

which is the symmetric part of the tensor product of two vectors. Thus $\hat{P}_{\Box\Box}$ projects to the symmetric part of the tensor product of two vectors. Also, note that

$$\begin{aligned}
(\hat{P}_{\Box\Box})^{i_1i_2}_{j_1j_2}(\hat{P}_{\Box\Box})^{j_1j_2}_{k_1k_2} &= \frac{1}{4}\left[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\right]\left[\delta^{j_1}_{k_1}\delta^{j_2}_{k_2} + \delta^{j_1}_{k_2}\delta^{j_2}_{k_1}\right]\\
&= \frac{1}{4}\left[\delta^{i_1}_{k_1}\delta^{i_2}_{k_2} + \delta^{i_1}_{k_2}\delta^{i_2}_{k_1} + \delta^{i_1}_{k_2}\delta^{i_2}_{k_1} + \delta^{i_1}_{k_1}\delta^{i_2}_{k_2}\right]\\
&= \frac{1}{2}\left[\delta^{i_1}_{k_1}\delta^{i_2}_{k_2} + \delta^{i_1}_{k_2}\delta^{i_2}_{k_1}\right]\\
&= (\hat{P}_{\Box\Box})^{i_1i_2}_{k_1k_2},
\end{aligned}$$

which is the property $\hat{P}^2 = \hat{P}$ with the correct placing of indices.

- $n = 2$ and $R = \genfrac{}{}{0pt}{}{\Box}{\Box}$

$$\begin{aligned}
(\hat{P}_{\genfrac{}{}{0pt}{}{\Box}{\Box}})^{i_1i_2}_{j_1j_2}v^{j_1}w^{j_2} &= \frac{1}{2}\left[\chi_{\genfrac{}{}{0pt}{}{\Box}{\Box}}(\mathbb{1})(\mathbb{1})^{i_1i_2}_{j_1j_2} + \chi_{\genfrac{}{}{0pt}{}{\Box}{\Box}}((1\ 2))((1\ 2))^{i_1i_2}_{j_1j_2}\right]v^{j_1}w^{j_2}\\
&= \frac{1}{2}\left[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\right]v^{j_1}w^{j_2}\\
&= \frac{1}{2}\left[v^{i_1}w^{i_2} - w^{i_1}v^{i_2}\right],
\end{aligned}$$

which is the antisymmetric part of the tensor product of two vectors. Thus $\hat{P}_{\genfrac{}{}{0pt}{}{\Box}{\Box}}$ projects to the antisymmetric part of the tensor product of two vectors. Also, note that

$$
\begin{aligned}
(\hat{P}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}})^{i_1 i_2}_{j_1 j_2}(\hat{P}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}})^{j_1 j_2}_{k_1 k_2} &= \frac{1}{4}\left[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\right]\left[\delta^{j_1}_{k_1}\delta^{j_2}_{k_2} - \delta^{j_1}_{k_2}\delta^{j_2}_{k_1}\right] \\
&= \frac{1}{4}\left[\delta^{i_1}_{k_1}\delta^{i_2}_{k_2} - \delta^{i_1}_{k_2}\delta^{i_2}_{k_1} - \delta^{i_1}_{k_2}\delta^{i_2}_{k_1} + \delta^{i_1}_{k_1}\delta^{i_2}_{k_2}\right] \\
&= \frac{1}{2}\left[\delta^{i_1}_{k_1}\delta^{i_2}_{k_2} - \delta^{i_1}_{k_2}\delta^{i_2}_{k_1}\right] \\
&= (\hat{P}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}})^{i_1 i_2}_{k_1 k_2},
\end{aligned}
$$

which, once again, is the property $\hat{P}^2 = \hat{P}$ with the correct placing of indices.

We also observe that these operators are complete:

$$
(\hat{P}_{\square\square} + \hat{P}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}})^I_J = \mathbb{1}^I_J.
$$

Recall earlier that the general form of the projection operator was given by $(\hat{P}_R)^I_J = \frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)(\sigma)^I_J$. We now wish to compute the product of two projection operators:

$$
\hat{P}_R\hat{P}_S = \frac{d_R}{n!}\frac{d_S}{n!}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\chi_R(\sigma_1)\chi_S(\sigma_2)\sigma_1\sigma_2,
$$

where everything on the right hand side of the equation is a number except for $\sigma_1\sigma_2$ which is a matrix, with the following property (if we consider the left action):

$$
(\sigma_1)^I_J(\sigma_2)^J_K = (\sigma_2\cdot\sigma_1)^I_K.
$$

We now change variables from σ_1,σ_2 to σ_1,$\psi = \sigma_2\cdot\sigma_1$, where σ_1 is fixed and ψ will run over the same S_n as σ_2. Therefore the multiplication becomes

$$
\hat{P}_R\hat{P}_S = \frac{d_R}{n!}\frac{d_S}{n!}\sum_{\sigma_1\in S_n}\sum_{\psi\in S_n}\chi_R(\sigma_1)\chi_S(\psi\cdot\sigma_1^{-1})(\psi)^I_K,
$$

since $\psi = \sigma_2\cdot\sigma_1$. Multiplying both sides to the right by σ_1^{-1} results in $\psi\cdot\sigma_1^{-1} = \sigma_2\cdot\sigma_1\sigma_1^{-1} = \sigma_2\cdot\mathbb{1} = \sigma_2$. The term $\chi_R(\sigma_1)\chi_S(\psi\cdot\sigma_1^{-1})$ suggests we use the Fundamental Orthogonality relation (see appendix A for a review of this relation). Thus, we obtain

$$
\begin{aligned}
\hat{P}_R\hat{P}_S &= \frac{d_R}{n!}\frac{d_S}{n!}\sum_{\psi\in S_n}\left(\sum_{\sigma_1\in S_n}(\Gamma_R(\sigma_1))_{ii}\Gamma_S(\psi)_{jk}\Gamma_S(\sigma_1^{-1})_{kj}\right)(\psi)^I_K \\
&= \delta_{RS}\frac{d_S}{n!}\sum_{\psi\in S_n}\chi_S(\psi)\psi \\
&= \delta_{RS}\hat{P}_S.
\end{aligned}
$$

Therefore, the general result of multiplying two projection operators, of irrep R and S respectively, is given as

$$
\hat{P}_R\hat{P}_S = \delta_{RS}\hat{P}_R,
$$

where $R \vdash n$.

Let V_N denote the N dimensional vector space. This space has a basis given by the N basis vectors

$$|e^1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad |e^2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad |e^N\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

where each vector has N components. This is a complex vector space so that the general vector in this case can be expressed as

$$|v\rangle = \sum_{i=1}^{N} c_i |e^i\rangle$$

with c_i are some complex numbers. The dual vector space is given by row vectors, with basis $\langle e_i|$. To get the dual to $|v\rangle$, we transpose columns into rows and complex conjugate the expansion coefficients

$$\langle v| = \sum_{i=1}^{N} c_i^* \langle e_i|.$$

The inner product is given by

$$\langle v|v\rangle = \sum_{i=1}^{N} |c_i|^2.$$

The group of $N \times N$ unitary matrices, $U(N)$, has the obvious action on this space, acting by matrix multiplication on vectors

$$|v\rangle \to U|v\rangle \qquad \langle v| \to \langle v|U^\dagger.$$

The inner product under the action of $U(N)$

$$\langle v|v\rangle \to \langle v|U^\dagger U|v\rangle = \langle v|v\rangle$$

is invariant. The tensor product of n copies of V_N vector spaces can be written as

$$V_N \otimes V_N \otimes \cdots \otimes V_N \equiv V_N^{\otimes n},$$

where $V_N^{\otimes n}$ is an N^n dimensional space. $V_N^{\otimes n}$ inherits a natural action of the unitary group from the action on V_N:

$$\begin{aligned}
&(U^{\otimes n})|v(1)\rangle \otimes |v(2)\rangle \otimes \cdots \otimes |v(n)\rangle = \\
&= (U \otimes U \otimes \cdots \otimes U)|v(1)\rangle \otimes |v(2)\rangle \otimes \cdots \otimes |v(n)\rangle \\
&= U|v(1)\rangle \otimes U|v(2)\rangle \otimes \cdots \otimes U|v(n)\rangle.
\end{aligned}$$

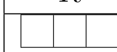
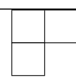
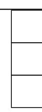
U acts in the same way (by matrix multiplication with the $N \times N$ unitary matrix representing U) on every tensor in the tensor product. One can view $U^{\otimes n}$ as an N^n dimensional matrix. We will now define an action of the symmetric group S_n on $V_N^{\otimes n}$. The symmetric group acts by interchanging the order of the vectors in the tensor product, without shuffling the components of any given vector

$$(\sigma)|v(1)\rangle \otimes |v(2)\rangle \otimes \cdots \otimes |v(n)\rangle = |v(\sigma(1))\rangle \otimes |v(\sigma(2))\rangle \otimes \cdots \otimes |v(\sigma(n))\rangle.$$

An important observation is the fact that the actions of the symmetric group S_n and the unitary group $U(N)$ on $V_N^{\otimes n}$ commute:

$$\begin{aligned} U^{\otimes n} \cdot \sigma \cdot (|v(1)\rangle \otimes |v(2)\rangle \otimes \cdots \otimes |v(n)\rangle) &= U^{\otimes n} \cdot (|v(\sigma(1))\rangle \otimes |v(\sigma(2))\rangle \otimes \cdots \otimes |v(\sigma(n))\rangle) \\ &= U|v(\sigma(1))\rangle \otimes U|v(\sigma(2))\rangle \otimes \cdots \otimes U|v(\sigma(n))\rangle \\ &= \sigma \cdot (U|v(1)\rangle \otimes U|v(2)\rangle \otimes \cdots \otimes U|v(n)\rangle) \\ &= \sigma \cdot U^{\otimes n} \cdot (|v(1)\rangle \otimes |v(2)\rangle \otimes \cdots \otimes |v(n)\rangle). \end{aligned}$$

The representations of $U(N)$ are labelled by Young diagrams with no more than N rows. We will denote the dimension of a $U(N)$ irrep by Dim_R. Consider $n = 3$, thus we work in the vector space $V_N^{\otimes 3}$, the table below displays the relation between R, d_R, and Dim_R

R	d_R	Dim_R
	1	$\frac{N(N+1)(N+2)}{6}$
	2	$\frac{N(N+1)(N-1)}{3}$
	1	$\frac{N(N-1)(N-2)}{6}$

Each Young-Yamanouchi diagram is a subspace. Summing up the products of d_R and Dim_R with their respective representations R, we see that

$$\begin{aligned} \sum_R d_R \mathrm{Dim}_R &= \frac{N(N+1)(N+2)}{6} + \frac{2N(N+1)(N-1)}{3} + \frac{N(N-1)(N-2)}{6} \\ &= N^3\left(\frac{1}{6} + \frac{2}{3} + \frac{1}{6}\right) + N^2\left(\frac{3}{6} - \frac{3}{6}\right) + N\left(\frac{2}{6} - \frac{2}{3} + \frac{2}{6}\right) \\ &= N^3 \end{aligned}$$

which is an insightful result since the dimension of $V_N^{\otimes 3}$ is N^3. In fact, $\sum_R d_R \mathrm{Dim}_R$ is equal to the total number of states in the vector space $V_N^{\otimes n}$ in general. This shows each state is labelled by a unique Young-Yamanouchi pattern and a unique Gelfand-Tsetlin pattern.

The Gelfand-Tsetlin pattern labelling chooses basis states that are simultaneous eigenstates of all the matrices $J_z^{(l)}$, and further, explicit formulas are known for the matrix elements of the $J_\pm^{(l)}$ with respect to these basis states. Thus the Gelfand-Tsetlin basis is also the natural extension of what we are used to from our studies of angular momentum in quantum mechanics.

We will now describe how the Gelfand-Tsetlin patterns are constructed. An inequivalent irreducible representation for $GL(N, \mathbb{C})$ is uniquely given by specifying the sequence of N integers

$$\mathbf{m} = (m_{1N}, m_{2N}, \ldots, m_{NN},) \tag{1}$$

satisfying $m_{kN} \geq m_{k+1,N}$ for $1 \leq k \leq N-1$. Notice that this sequence can be identified with the row lengths of a Young diagram R that has no more than N rows - this is the Young diagram labelling the $GL(N, \mathbb{C})$ irreducible representation. We call the sequence (1) the weight of the irreducible representation. The restriction of this irrep onto the subgroup $GL(N-1, \mathbb{C})$ is reducible. It decomposes into a direct sum of $GL(N-1, \mathbb{C})$ irreps with highest weights

$$\mathbf{m}' = (m_{1,N-1}, m_{2,N-1}, \ldots, m_{N-1,N-1}),$$

for which the "betweenness" conditions

$$m_{kN} \geq m_{k,N-1} \geq m_{k+1,N} \quad \text{for} \quad 1 \leq k \leq N-1$$

hold. This specifies the branching rule for how a $GL(N, \mathbb{C})$ irrep decomposes when we restrict to the $GL(N-1, \mathbb{C})$ subgroup. We can repeat this procedure until we get to $GL(1, \mathbb{C})$ which has one-dimensional carrier spaces. The Gelfand-Tsetlin labelling assembles this sequence of representations of the subgroups into a Gelfand-Tsetlin pattern. There is a unique pattern for each state in the basis of the carrier space of the original $GL(N, \mathbb{C})$ irrep. The pattern can be written as a triangular arrangement of integers, denoted M, with the structure

$$M = \begin{bmatrix} m_{1N} & & m_{2N} & & \ldots & & m_{N-1,N} & & m_{NN} \\ & m_{1,N-1} & & m_{2,N-1} & & \ldots & & m_{N-1,N-1} & \\ & & \ldots & & \ldots & & \ldots & & \\ & & m_{12} & & & & m_{22} & & \\ & & & & m_{11} & & & & \end{bmatrix}$$

The top row contains the weight that species the irrep of the state and the entries of lower rows are subject to the betweenness condition. The lower rows give the sequence of irreps our state belongs to as we pass through successive restrictions from $GL(N, \mathbb{C})$ to $GL(N-1, \mathbb{C})$ to $\ldots$ to $GL(1, \mathbb{C})$. For each possible pattern we have a distinct vector. The vectors labelled by two distinct patterns are orthogonal and the dimension of the irrep is given by the total number of distinct Gelfand-Tsetlin patterns that can be constructed. As we have already described above, everything we have said about $GL(N, \mathbb{C})$ applies, without any change, to $U(N)$. This orthogonal basis of $U(N)$ is called the Gelfand-Tsetlin basis.

2.4.2 Equivalence relation and equivalence classes

$\sim$ is an equivalence relation if it obeys the following three conditions:

- $a \sim a$ (reflectivity)
- $a \sim b$ then $b \sim a$ (symmetry)
- $a \sim b$ and $b \sim c$, then it implies that $a \sim c$ (transitivity).

Now, we say g is conjugate to h, written as $g \sim h$ with g,$h \in \mathcal{G}$, if $g = \sigma h \sigma^{-1}$ for some $\sigma \in \mathcal{G}$. We now make and prove the statement that "conjugate to" is an equivalence relation.

<u>Proof:</u>

- Reflexivity is satisfied with the condition $g = \mathbb{1} g \mathbb{1}^{-1}$

- Symmetry is satisfied when $g = \sigma h \sigma^{-1}$ implies $h = \sigma^{-1} g \sigma = \sigma^{-1} g (\sigma^{-1})^{-1}$
- Transitivity is satisfied when we let $g = \sigma_1 h \sigma_1^{-1}$ and $h = \sigma_2 j \sigma_2^{-1}$ which implies $g = \sigma_1(\sigma_2 j \sigma_2^{-1})\sigma_1^{-1} = \sigma_1\sigma_2 j(\sigma_1\sigma_2)^{-1}$ $\square$

This equivalence relation partitions the group into *conjugacy classes* (this is the name of these equivalence classes). Suppose we now have a look at the characters of g and h, denoted as $\chi_R(g) = \text{Tr}(\Gamma_R(g))$ and $\chi_R(h) = \text{Tr}(\Gamma_R(h))$ respectively, with $g \sim h$. Then

$$\begin{aligned}
\chi_R(h) = \text{Tr}(\Gamma_R(h)) &= \text{Tr}(\Gamma_R(\sigma)\Gamma_R(g)\Gamma_R(\sigma^{-1})) \\
&= \text{Tr}(\Gamma_R(\sigma^{-1})\Gamma_R(\sigma)\Gamma_R(g)) \\
&= \text{Tr}(\Gamma_R(\mathbb{1})\Gamma_R(g)) \\
&= \text{Tr}(\Gamma_R(g)) \\
&= \chi_R(g),
\end{aligned}$$

where in the second line the cyclicity of the trace is used, and in the third line the property $\Gamma_R(a) \cdot \Gamma_R(b) = \Gamma_R(a \cdot b)$ for a,$b \in \mathcal{G}$ and the fact that $\sigma^{-1}\sigma = \mathbb{1}$ for $\sigma \in \mathcal{G}$ was used. The result $(\chi_R(g) = \chi_R(h))$ tells us that all elements in a conjugacy class have the same character. Now, we have $R \vdash n$, which means $\hat{P}_R$ lives in $V_N^{\otimes n}$, and we wish to evaluate the product on $\hat{P}_R$ and ψ (where $\psi \in S_n$). For the representation of S_n on $V^{\otimes n}$ recall that $(\sigma_1)^I_J(\sigma_2)^J_K = (\sigma_2 \cdot \sigma_1)^I_K$. The multiplication is given as

$$(\hat{P}_R)^I_J(\psi)^J_K = \frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)(\sigma)^I_J(\psi)^J_K.$$

By changing the summation variable from σ to τ, where

$$\begin{aligned}
(\sigma)^I_J(\psi)^J_K &= (\psi)^I_J(\tau)^J_K \\
\Rightarrow (\sigma)^I_L &= (\psi)^I_J(\tau)^J_K(\psi^{-1})^K_L \\
&= (\psi^{-1}\tau\psi)^I_L,
\end{aligned}$$

where the right action of the group was used. Then

$$\begin{aligned}
(\hat{P}_R)^I_J(\psi)^J_K &= \frac{d_R}{n!}\sum_{\tau\in S_n}\chi_R(\psi^{-1}\tau\psi)(\psi)^I_Q(\tau)^Q_L(\psi^{-1})^L_J(\psi)^J_K \\
&= \frac{d_R}{n!}\sum_{\tau\in S_n}\chi_R(\psi^{-1}\tau\psi)(\psi)^I_Q(\tau)^Q_L \\
&= (\psi)^I_Q(\hat{P}_R)^Q_L.
\end{aligned}$$

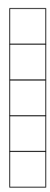
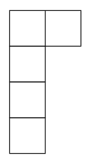
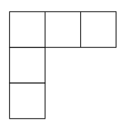
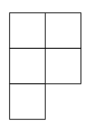
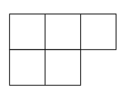
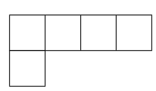
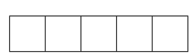
From this result, we conclude that $\hat{P}_R\psi = \psi\hat{P}_R$, so $\hat{P}_R$ and ψ commute with each other, i.e $[\hat{P}_R, \psi] = 0$.

It should be noted that conjugacy classes of the symmetric group correspond to the cycle structure.

For example, the S_5 group has the following cycle structures

1. (1)(2)(3)(4)(5)
2. (1 2)(3)(4)(5)
3. (1 2 3)(4)(5)
4. (1 2)(3 4)(5)
5. (1 2 3)(4 5)
6. (1 2 3 4)(5)
7. (1 2 3 4 5)

and they correspond to the following Young diagrams

1. 
2. 
3. 
4. 
5. 
6. 
7. 

From this correspondence, we conclude that Young diagrams also label conjugacy classes!

2.4.3 Schur Polynomials

We will now introduce the Schur Polynomial [8],[9] which is given as

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)(\sigma)^I_J (Z)^J_I,$$

where Z is a $N \times N$ complex matrix. Consider the example that $R = \square\square\square$ and $n = 3$, then

$$\begin{aligned}
\chi_{\square\square\square}(Z) &= \frac{1}{3!}(\chi_{\square\square\square}(\mathbb{1})Z^{i_1}_{i_i}Z^{i_2}_{i_2}Z^{i_3}_{i_3} + \chi_{\square\square\square}((1\ 2))Z^{i_1}_{i_2}Z^{i_2}_{i_1}Z^{i_3}_{i_3} + \chi_{\square\square\square}((1\ 3))Z^{i_1}_{i_3}Z^{i_2}_{i_2}Z^{i_3}_{i_1} \\
&\quad + \chi_{\square\square\square}((2\ 3))Z^{i_1}_{i_1}Z^{i_2}_{i_3}Z^{i_3}_{i_2} + \chi_{\square\square\square}((1\ 2\ 3))Z^{i_1}_{i_2}Z^{i_2}_{i_3}Z^{i_3}_{i_1} + \chi_{\square\square\square}((1\ 3\ 2))Z^{i_1}_{i_3}Z^{i_2}_{i_1}Z^{i_3}_{i_2}) \\
&= \frac{1}{6}(Z^{i_1}_{i_i}Z^{i_2}_{i_2}Z^{i_3}_{i_3} + Z^{i_1}_{i_2}Z^{i_2}_{i_1}Z^{i_3}_{i_3} + Z^{i_1}_{i_3}Z^{i_2}_{i_2}Z^{i_3}_{i_1} + Z^{i_1}_{i_1}Z^{i_2}_{i_3}Z^{i_3}_{i_2} + Z^{i_1}_{i_2}Z^{i_2}_{i_3}Z^{i_3}_{i_1} + Z^{i_1}_{i_3}Z^{i_2}_{i_1}Z^{i_3}_{i_2}) \\
&= \frac{1}{6}(\operatorname{Tr}(Z)^3 + 3\operatorname{Tr}(Z^2)\operatorname{Tr}(Z) + 2\operatorname{Tr}(Z^3)).
\end{aligned}$$

The relation (for $n = 3$) between the class, the permutation σ and the trace $\operatorname{Tr}(\sigma Z^{\otimes 3})$ is given below

Class	σ	$\operatorname{Tr}(\sigma Z^{\otimes 3})$
1^3	$\mathbb{1}$	$\operatorname{Tr}(Z)^3$
$2\ 1$	$(1\ 2), (1\ 3), (2\ 3)$	$\operatorname{Tr}(Z)\operatorname{Tr}(Z^2)$
3	$(1\ 2\ 3), (1\ 3\ 2)$	$\operatorname{Tr}(Z^3)$

We thus conclude from this that conjugacy classes of the symmetric group correspond to trace (physical) operators. Recall that taking a trace produces a gauge invariant. We can rewrite the Schur Polynomial as [8],[9]

$$\chi_R(Z) = \frac{1}{d_R}\left[\frac{d_R}{n!}\sum_{\sigma \in S_n}\chi_R(\sigma)(\sigma)^I_J(Z)^J_I\right] = \frac{1}{d_R}\operatorname{Tr}(\hat{P}_R Z^{\otimes n}).$$

We now wish to evaluate the 2-point function of the Schur Polynomial. The correlator is given by [8],[9]

$$\langle \chi_R(Z)\chi^\dagger_S(Z)\rangle = \frac{1}{d_R d_S}(\hat{P}_R)^I_J(\hat{P}_S)^K_L\langle (Z^{\otimes n})^J_I(Z^{\dagger^{\otimes n}})^L_K\rangle,$$

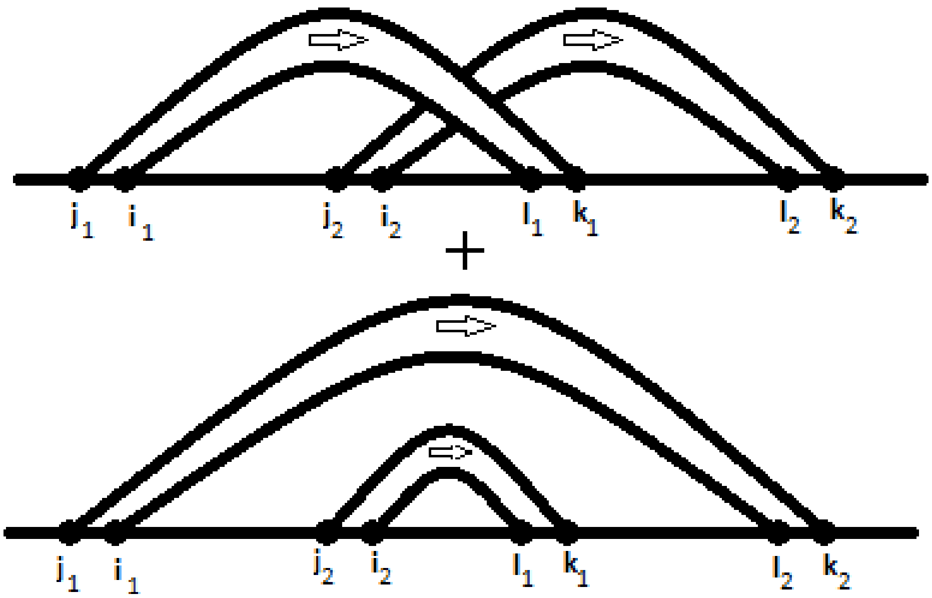
where $(Z^{\otimes n})^I_J = Z^{i_1}_{j_1}\dots Z^{i_n}_{j_n}$ and

$$\langle (Z^{\otimes n})^J_I(Z^{\dagger^{\otimes n}})^L_K\rangle = \sum_{\sigma \in S_n}(\sigma^{-1})^I_L(\sigma)^K_J$$

where the summation in this case is a sum over all ribbons graphs resulting in $n!$ diagrams since the permutation is an element of S_n. To illustrate this statement, let's take $n = 2$ for example. In this case

$$\begin{aligned}
\langle Z^{j_1}_{i_1}Z^{j_2}_{i_2}Z^{\dagger l_1}_{k_1}Z^{\dagger l_2}_{k_2}\rangle &= (\mathbb{1})^I_L(\mathbb{1})^K_J + ((1\ 2))^I_L((1\ 2))^K_J \\
&= \delta^{i_1}_{l_1}\delta^{i_2}_{l_2}\delta^{k_1}_{j_1}\delta^{k_2}_{j_2} + \delta^{i_1}_{l_2}\delta^{i_2}_{l_1}\delta^{k_1}_{j_2}\delta^{k_2}_{j_1},
\end{aligned}$$

which is indeed the result illustrated in the pairing below to form ribbon diagrams:



Therefore, the 2-point correlator of the Schur polynomials is [8],[9]

$$\begin{aligned}
\langle \chi_R(Z)\chi_S^\dagger(Z)\rangle &= \frac{1}{d_R d_S}\sum_{\sigma\in S_n}(\hat{P}_R)^I_J(\hat{P}_S)^K_L(\sigma^{-1})^I_L(\sigma)^K_J \\
&= \frac{1}{d_R d_S}\sum_{\sigma\in S_n}\mathrm{Tr}\left(\hat{P}_R\sigma^{-1}\hat{P}_S\sigma\right) \\
&= \frac{1}{d_R d_S}\sum_{\sigma\in S_n}\mathrm{Tr}\left(\hat{P}_R\hat{P}_S\right), \qquad \because [\hat{P}_S,\sigma]=0 \ \&\ \sigma^{-1}\cdot\sigma = \mathbb{1} \\
&= \frac{n!}{d_R d_S}\mathrm{Tr}\left(\hat{P}_R\hat{P}_S\right) \\
&= \frac{n!\delta_{RS}}{d_R d_S}\mathrm{Tr}\left(\hat{P}_R\right), \qquad \because \hat{P}_R\hat{P}_S = \delta_{RS}\hat{P}_R \\
&= \frac{n!\delta_{RS}}{d_R d_S} d_R Dim_R, \qquad \because \mathrm{Tr}\,(\hat{P}_R) = d_R Dim_R \\
&= \frac{n!\delta_{RS}}{d_R} Dim_R \\
&= \delta_{RS}\frac{n!}{\left(\frac{n!}{\mathrm{hooks}_R}\right)}\frac{\mathcal{F}_R}{\mathrm{hooks}_R} \\
&= \delta_{RS}\mathcal{F}_R,
\end{aligned}$$

where $d_R = \frac{n!}{\mathrm{hooks}_R}$ and $\mathrm{Dim}_R = \frac{\mathcal{F}_R}{\mathrm{hooks}_R}$, where $\mathcal{F}_R$ denotes the product of the weights of boxes in the Young diagram R.

2.5 Two matrices Z and Y

Now, we will consider not just a single matrix Z but two matrices Z and Y, and we use the following results:

$$\langle Z_{ij}Z^\dagger_{kl}\rangle = \delta_{il}\delta_{jk}, \quad \langle Y_{ij}Y^\dagger_{kl}\rangle = \delta_{il}\delta_{jk}, \quad \langle Y_{ij}Z_{kl}\rangle = 0, \quad \langle Y_{ij}Z^\dagger_{kl}\rangle = 0.$$

Extremal correlation functions are correlation functions that only comprise of a collection of a single matrix (before that would be matrix Z). Extremal correlation functions get no quantum corrections. In general, the Z,Y correlators do receive quantum corrections. Suppose we have two Z's and two Y's. The observables we can construct are

$$\mathrm{Tr}(Z^2Y^2),\ \ \mathrm{Tr}(ZYZY),\ \ \mathrm{Tr}(Z^2Y)\,\mathrm{Tr}(Y),\ \ \mathrm{Tr}(Y^2Z)\,\mathrm{Tr}(Z),\ \ \mathrm{Tr}(Z^2)\,\mathrm{Tr}(Y^2),\ \ \mathrm{Tr}(ZY)^2,\ \ \mathrm{Tr}(Z^2)\,\mathrm{Tr}(Y)^2,$$
$$\mathrm{Tr}(Y^2)\,\mathrm{Tr}(Z)^2,\ \ \mathrm{Tr}(ZY)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y),\ \ \&\ \ \mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y)^2,$$

whereas if we have four Z's the observables we find are

$$\mathrm{Tr}(Z^4),\ \ \mathrm{Tr}(Z^3)\,\mathrm{Tr}(Z),\ \ \mathrm{Tr}(Z^2)\,\mathrm{Tr}(Z^2),\ \ \mathrm{Tr}(Z^2)\,\mathrm{Tr}(Z)^2,\ \ \&\ \ \mathrm{Tr}(Z)^4.$$

Thus, we conclude there will be more observables for two matrices than there would be for a single matrix. This is because Z and Y do not commute, i.e. $ZY \neq YZ$, in general. For operators constructed from nZ's and mY's, we will work in $V_N^{\otimes n+m}$. The tensor product of these matrices is

$$(Z^{\otimes n}Y^{\otimes m})^I_J \equiv Z^{i_1}_{j_1}\dots Z^{i_n}_{j_n}Y^{i_{n+1}}_{j_{n+1}}\dots Y^{i_{n+m}}_{j_{n+m}}.$$

The Bose symmetry associated to these matrices is given by

$$(\sigma^{-1})^I_J(Z^{\otimes n}Y^{\otimes m})^J_K(\sigma)^K_L = (Z^{\otimes n}Y^{\otimes m})^I_L, \quad \sigma \in S_n \times S_m.$$

In the general case, the observables are given as

$$\begin{aligned}\mathrm{Tr}(\rho Z^{\otimes n}Y^{\otimes m}) &= \mathrm{Tr}(\rho\sigma^{-1}Z^{\otimes n}Y^{\otimes m}\sigma)\\ &= \mathrm{Tr}(\sigma\rho\sigma^{-1}Z^{\otimes n}Y^{\otimes m})\\ &= (\sigma^{-1}\rho\sigma)^I_J(Z^{\otimes n}Y^{\otimes m})^J_I,\end{aligned}$$

since ρ and $\sigma^{-1}\rho\sigma$ correspond to the same physical observable for any $\sigma \in S_n \times S_m$. We say that ρ and τ are *restricted conjugate* if and only if

$$\rho = \sigma^{-1}\tau\sigma$$

for some $\sigma \in S_n \times S_m$. We will now prove the statement that "restricted conjugate" is an equivalence relation:
<u>Proof:</u>

- Reflexivity: $\rho = \mathbb{1}^{-1}\rho\mathbb{1}$, $\mathbb{1} \in S_n \times S_m$.
- Symmetry: $\rho = \sigma^{-1}\tau\sigma \Rightarrow \tau = \sigma\rho\sigma^{-1} = (\sigma^{-1})^{-1}\rho\sigma^{-1}$, $\sigma^{-1} \in S_n \times S_m$.
- Transitivity: $\rho = \sigma_1^{-1}\tau\sigma_1$ and $\tau = \sigma_2^{-1}\beta\sigma_2$, then

$$\rho = \sigma_1^{-1}\tau\sigma_1 = \sigma_1^{-1}\sigma_2^{-1}\beta\sigma_2\sigma_1 = (\sigma_2\sigma_1)^{-1}\beta\sigma_2\sigma_1,$$

  with $\sigma_2 \cdot \sigma_1 \in S_n \times S_m$. $\square$

Therefore restricted conjugate is an equivalence relation and thus we have a notion of *restricted conjugacy classes*. It is a highly non-trivial fact that the number of restricted conjugacy classes is equal to the number of independent gauge invariant operators that can be defined.

2.5.1 Intertwining map

An intertwining map is a generalised projection operator in $V^{\otimes\, n+m}$. It has many of the properties of a projector ($P_A P_B = \delta_{AB} P_{\tilde{A}}$ with A possibly standing for a collection of indices) but does not in general square to itself. They are important in the study of (restricted) Schur polynomials.

Recall for the one matrix observables, we had

$$\chi_R(Z) = \frac{1}{n!}\sum_{\sigma\in S_n} \chi_R(\sigma)(\sigma)^I_J (Z)^J_I = \frac{1}{n!}\sum_{\sigma\in S_n} \chi_R \,\mathrm{Tr}(\sigma Z^{\otimes n}) = \mathrm{Tr}(\hat{P}_R Z^{\otimes n}),$$

where $\hat{P}_R = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma$ and σ is the previously introduced action on $V^{\otimes n+m}$. Also recall that for the one matrix problem, the correlators are

$$\langle (Z^{\otimes n})^I_J (Z^{\dagger^{\otimes n}})^K_L\rangle = \sum_{\sigma\in S_n} (\sigma^{-1})^I_L (\sigma)^K_J$$

and

$$\begin{aligned}
\langle \chi_R(Z)\chi^\dagger_S(Z)\rangle &= (\hat{P}_R)^I_J (\hat{P}_S)^K_L \langle (Z^{\otimes n})^J_I (Z^{\dagger^{\otimes n}})^J_I\rangle \\
&= \sum_{\sigma\in S_n} (\hat{P}_R)^I_J (\hat{P}_S)^K_L (\sigma^{-1})^J_K (\sigma)^L_I \\
&= \sum_{\sigma\in S_n} \mathrm{Tr}(\hat{P}_R \sigma \hat{P}_S \sigma^{-1}) \\
&= \sum_{\sigma\in S_n} \mathrm{Tr}(\hat{P}_R \hat{P}_S) \\
&= n!\ \mathrm{Tr}(\hat{P}_R \hat{P}_S).
\end{aligned}$$

We now wish to introduce a "projection operator" for the case of two matrices. We denote this operator as $P_{R,(r,s)\alpha\beta}$ where $R \vdash n+m$ and labels irreps of S_{n+m}, $r \vdash n$, $s \vdash m$, (r,s) labels irreps of $S_n \times S_m$, and α and β are multiplicity labels that label different copies of the same subgroup irrep. Suppose we consider $R = \yng(3,2,1)$ then $\dim_{\yng(3,2,1)} = \frac{6!}{45} = 16$. Now remove m (which in this case is equal to 3) boxes such that you are still left with a valid Young diagram. In our example for R, the combination of Young diagrams we get after removing $m(=3)$ boxes are

$$\left(\yng(2,1)\,,\ \yng(1)\times\yng(1)\times\yng(1)\right),\ \left(\yng(1,1,1)\,,\ \yng(2,1)\right),\ \&\ \left(\yng(3)\,,\ \yng(2,1)\right),$$

where the last two diagrams, the diagrams on the right are diagrams that came about due to sides being shared between boxes that are removed. The first combination can further be written as

$$\left(\ydiagram{2,1}\,,\ \ydiagram{1}\times\ydiagram{1}\times\ydiagram{1}\right)=\left(\ydiagram{2,1}\,,\ \ydiagram{3}\right)+\left(\ydiagram{2,1}\,,\ \ydiagram{2,1}\right)$$
$$+\left(\ydiagram{2,1}\,,\ \ydiagram{2,1}\right)+\left(\ydiagram{2,1}\,,\ \ydiagram{1,1,1}\right),$$

since $\ydiagram{1}\times\ydiagram{1}\times\ydiagram{1}=\ydiagram{1}\times\left(\ydiagram{2}\oplus\ydiagram{1,1}\right)=\ydiagram{3}+\ydiagram{2,1}+\ydiagram{2,1}+\ydiagram{1,1,1}$. By adding up the dimensions of the combinations, we get

$$\begin{aligned}
&\dim\left(\ydiagram{2,1}\,,\ \ydiagram{1}\times\ydiagram{1}\times\ydiagram{1}\right)+\dim\left(\ydiagram{1,1,1}\,,\ \ydiagram{2,1}\right)+\dim\left(\ydiagram{3}\,,\ \ydiagram{2,1}\right)\\
&=\dim\left(\ydiagram{2,1}\,,\ \ydiagram{3}\right)+\dim\left(\ydiagram{2,1}\,,\ \ydiagram{2,1}\right)+\dim\left(\ydiagram{2,1}\,,\ \ydiagram{2,1}\right)+\dim\left(\ydiagram{2,1}\,,\ \ydiagram{1,1,1}\right)+\dim\left(\ydiagram{1,1,1}\,,\ \ydiagram{2,1}\right)+\dim\left(\ydiagram{3}\,,\ \ydiagram{2,1}\right)\\
&=((2\times 1)+(2\times 2)+(2\times 2)+(2\times 1))+(1\times 2)+(1\times 2)\\
&=2+4+4+2+2+2\\
&=16
\end{aligned}$$

which is the dimension of $R=\ydiagram{3,2,1}$. Note that there are two combinations that are $\left(\ydiagram{2,1}\,,\ \ydiagram{2,1}\right)$.

We introduce a number g_{Rrs} called the Littlewood-Richardson number to count this multiplicity. In our case $g_{\ydiagram{3,2,1}\,\ydiagram{2,1}\,\ydiagram{2,1}}=2$. In what follows we will use multiplicity labels $\alpha,\beta=1\dots g_{Rrs}$. We need to introduce one more concept: $\chi_{R,(r,s)\alpha\beta}(\sigma)$. This is the *restricted character* and it is given by

$$\chi_{R,(r,s)\alpha\beta}(\sigma)=\sum_{i=1}^{d_r d_s}\langle R,(r,s)\alpha;i|\Gamma_R(\sigma)|R,(r,s)\beta;i\rangle,$$

where $\sigma\in S_{n+m}$, i denotes the state, $R,(r,s)\alpha$ and $R,(r,s)\beta$ denote the representation, $R,(r,s)\alpha;i$ denotes the row index, and $R,(r,s)\beta;i$ denotes the column index. $P_{R,(r,s)\alpha\beta}$ is now given by

$$P_{R,(r,s)\alpha\beta}\equiv\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}(\sigma)\sigma$$

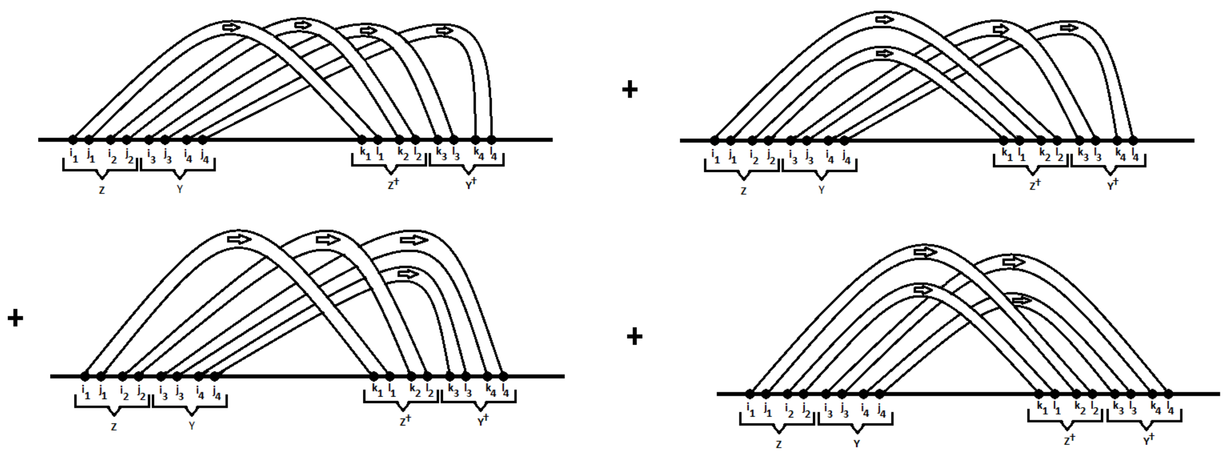
where σ uses the familiar action on $V^{\otimes n+m}$. $P_{R,(r,s)\alpha\beta}$ is called the *intertwining map* and it maps you between the α and β copy of (r,s). Note that for 2 matrices Z and Y, the 2-point function is given as

$$\langle (Z^{\otimes n} Y^{\otimes m})^I_J (Z^{\dagger^{\otimes n}} Y^{\dagger^{\otimes m}})^K_L \rangle = \sum_{\sigma \in S_n \times S_m} (\sigma)^I_L (\sigma^{-1})^K_J.$$

To illustrate this statement, let's take $n = 2$ and $m = 2$ for example, then

$$\begin{aligned}
\langle Z^{i_1}_{j_1} Z^{i_2}_{j_2} Y^{i_3}_{j_3} Y^{i_4}_{j_4} Z^{\dagger k_1}_{l_1} Z^{\dagger k_2}_{l_2} Y^{\dagger k_3}_{l_3} Y^{\dagger k_4}_{l_4} \rangle &= (\mathbb{1})^I_L (\mathbb{1})^K_J + ((1\ 2))^I_L ((1\ 2))^K_J \\
&+ ((3\ 4))^I_L ((3\ 4))^K_J + ((1\ 2)(3\ 4))^I_L ((1\ 2)(3\ 4))^K_J \\
&= \delta^{i_1}_{l_1} \delta^{i_2}_{l_2} \delta^{i_3}_{l_3} \delta^{i_4}_{l_4} \delta^{j_1}_{k_1} \delta^{j_2}_{k_2} \delta^{j_3}_{k_3} \delta^{j_4}_{k_4} + \delta^{i_1}_{l_2} \delta^{i_2}_{l_1} \delta^{i_3}_{l_3} \delta^{i_4}_{l_4} \delta^{j_1}_{k_2} \delta^{j_2}_{k_1} \delta^{j_3}_{k_3} \delta^{j_4}_{k_4} \\
&+ \delta^{i_1}_{l_1} \delta^{i_2}_{l_2} \delta^{i_3}_{l_4} \delta^{i_4}_{l_3} \delta^{j_1}_{k_1} \delta^{j_2}_{k_2} \delta^{j_3}_{k_4} \delta^{j_4}_{k_3} + \delta^{i_1}_{l_2} \delta^{i_2}_{l_1} \delta^{i_3}_{l_4} \delta^{i_4}_{l_3} \delta^{j_1}_{k_2} \delta^{j_2}_{k_1} \delta^{j_3}_{k_4} \delta^{j_4}_{k_3},
\end{aligned}$$

which is indeed the result seen in the pairing seen below to form ribbon diagrams:



The restricted Schur Polynomial for 2 matrices Z and Y is given as [1],[2]

$$\chi_{R,(r,s)\alpha\beta}(Z, Y) \equiv \mathrm{Tr}(P_{R,(r,s)\alpha\beta} Z^{\otimes n} Y^{\otimes m}),$$

and the 2-point function is given as [1],[2]

$$\begin{aligned}
\langle \chi_{R,(r,s)\alpha\beta}(Z, Y) \chi_{T,(t,u)\gamma\delta}(Z^\dagger, Y^\dagger) \rangle &= \langle \mathrm{Tr}(P_{R,(r,s)\alpha\beta} Z^{\otimes n} Y^{\otimes m}) \, \mathrm{Tr}(P_{T,(t,u)\gamma\delta} Z^{\dagger^{\otimes n}} Y^{\dagger^{\otimes m}}) \rangle \\
&= (P_{R,(r,s)\alpha\beta})^I_J (P_{T,(t,u)\gamma\delta})^K_L \langle (Z^{\otimes n} Y^{\otimes m})^J_I (Z^{\dagger^{\otimes n}} Y^{\dagger^{\otimes m}})^L_K \rangle \\
&= \sum_{\sigma \in S_n \times S_m} (P_{R,(r,s)\alpha\beta})^I_J (P_{T,(t,u)\gamma\delta})^K_L (\sigma)^J_K (\sigma^{-1})^L_I \\
&= \sum_{\sigma \in S_n \times S_m} \mathrm{Tr}(P_{R,(r,s)\alpha\beta} \ \sigma P_{T,(t,u)\gamma\delta} \ \sigma^{-1}) \\
&= n!m! \ \mathrm{Tr}(P_{R,(r,s)\alpha\beta} P_{T,(t,u)\gamma\delta}).
\end{aligned}$$

We should note that the intertwining map $P_{R,(r,s)\alpha\beta}$ has properties that are similar to the properties of a projection operator. These properties are [1],[2]:

- $P_{R,(r,s)\alpha\beta} \cdot P_{T,(t,u)\gamma\delta} = \delta_{RT} \delta_{rt} \delta_{su} \delta_{\beta\gamma} \# P_{R,(r,s)\alpha\delta}$

- $P_{R,(r,s)\alpha\alpha} \cdot P_{R,(r,s)\alpha\alpha} = \# P_{R,(r,s)\alpha\alpha}$
- $[P_{R,(r,s)\alpha\beta}, \sigma] = 0, \quad \sigma \in S_n \times S_m$
- $\Gamma_{(r,s)\alpha}(\sigma) P_{R,(r,s)\alpha\beta} = P_{R,(r,s)\alpha\beta} \Gamma_{(r,s)\beta}(\sigma), \quad \sigma \in S_n \times S_m.$

Using these properties, we learn that [1],[2]

$$\langle \chi_{R,(r,s)\alpha\beta}(Z,Y) \chi^{\dagger}_{T,(t,u)\gamma\delta}(Z,Y) \rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\beta\gamma} \frac{\mathcal{F}_R \, \mathrm{hooks}_R}{\mathrm{hooks}_r \;\; \mathrm{hooks}_s}.$$

2.6 Dilatation Operator in the SU(2) sector

The (one loop) dilatation operator in the $SU(2)$ sector of $\mathcal{N} = 4$ super Yang Mills theory is given as [10]

$$D = -g^2_{YM} \, \mathrm{Tr}[Y, Z][\partial_Y, \partial_Z],$$

where g_{YM} is the coupling in the Yang Mills theory, $Y = \phi_3 + i\phi_4$ and $Z = \phi_1 + i\phi_2$, and ϕ^i_{ab} are scalar fields. When the dilatation operator acts on a restricted Schur Polynomial, we obtain [1],[2],[10]

$$D\chi_{R,(r,s)\alpha\beta} = \frac{g^2_{YM}}{(n-1)!(m-1)!} \sum_{\psi \in S_{n+m}} \mathrm{Tr}_{(r,s)\alpha\beta}(\Gamma_R((1,m+1)\psi - \psi(1,m+1)) \times$$
$$\times \delta^{i_1}_{i_{\psi(1)}} Y^{i_2}_{i_{\psi(2)}} \dots Y^{i_m}_{i_{\psi(m)}} (YZ - ZY)^{i_{m+1}}_{i_{\psi(m+1)}} Z^{i_{m+2}}_{i_{\psi(m+2)}} \dots Z^{i_{n+m}}_{i_{\psi(n+m)}}.$$

The consequence of $\delta^{i_1}_{i_{\psi(1)}}$ appearing in the summand is the sum over ψ runs only over permutations for which $\psi(1) = 1$. For us to perform the sum over ψ, we write the sum over S_{n+m} as a sum over cosets of the S_{n+m-1} subgroup that is obtained by keeping the permutations that satisfy $\psi(1) = 1$. After using the reduction rule for Schur Polynomials, our result becomes [1],[2],[10]

$$D\chi_{R,(r,s)\alpha\beta} = \frac{g^2_{YM}}{(n-1)!(m-1)!} \sum_{\psi \in S_{n+m-1}} \sum_{R'} c_{RR'} \, \mathrm{Tr}_{(r,s)\alpha\beta} \Big(\Gamma_R((1,m+1)) \Gamma_{R'}(\psi)$$
$$- \Gamma_{R'}(\psi) \Gamma_R((1,m+1)) \Big) Y^{i_2}_{i_{\psi(2)}} \dots Y^{i_m}_{i_{\psi(m)}} (YZ - ZY)^{i_{m+1}}_{i_{\psi(m+1)}} Z^{i_{m+2}}_{i_{\psi(m+2)}} \dots Z^{i_{n+m}}_{i_{\psi(n+m)}}.$$

The sum over irrep R' runs over all Young diagrams that can be obtained from irrep R when we drop a single box, and $c_{RR'}$ is the factor of the box that must be removed from R to get R'. The permutation $\Gamma((1,m+1))$ is not an element of the $S_n \times S_m$ subgroup; it mixes indices belonging to Zs and indices belonging to Ys. The dilatation operator has derivatives with respect to Z and Y in the same trace and they naturally mix Zs and Ys. We will make use of the following notation

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = Z^{i_1}_{i_{\sigma(1)}} \dots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} \dots Y^{i_{n+m}}_{i_{\sigma(n+m)}}.$$

Using the identities (with the fact that $\psi(1) = 1$)

$$Y^{i_2}_{i_{\psi(2)}} \dots Y^{i_m}_{i_{\psi(m)}} (YZ - ZY)^{i_{m+1}}_{i_{\psi(m+1)}} Z^{i_{m+2}}_{i_{\psi(m+2)}} \dots Z^{i_{n+m}}_{i_{\psi(n+m)}} = \mathrm{Tr}\left(\Big((1,m+1)\psi - \psi(1,m+1) \Big) Z^{\otimes n} Y^{\otimes m} \right)$$

and

$$\mathrm{Tr}(\sigma Z^{\otimes n}Y^{\otimes m}) = \sum_{T,(t,u)\gamma\delta} \frac{d_T n!m!}{d_t d_u (n+m)!} \mathrm{Tr}_{(t,u)\gamma\delta}(\Gamma_T(\sigma^{-1}))\chi_{T,(t,u)\gamma\delta}(Z,Y)$$

we obtain

$$D\chi_{R,(r,s)\alpha\beta}(Z,Y) = \sum_{T,(t,u)\gamma\delta} M_{R,(r,s)\alpha\beta;T,(t,u)\gamma\delta}\ \chi_{T,(t,u)\gamma\delta}(Z,Y),$$

where

$$M_{R,(r,s)\alpha\beta;T,(t,u)\gamma\delta} = g_{YM}^2 \sum_{\psi\in S_{n+m-1}} \sum_{R'} \frac{c_{RR'}d_T nm}{d_t d_u (n+m)!} \mathrm{Tr}_{(r,s)\alpha\beta}\Big(\Gamma_R((1,m+1))\Gamma_{R'}(\psi) - \Gamma_{R'}\Gamma_R((1,m+1))(\psi)\Big)\times$$
$$\times\ \mathrm{Tr}_{(t,u)\gamma\delta}\Big(\Gamma_{T'}(\psi^{-1})\Gamma_T((1,m+1)) - \Gamma_T((1,m+1))\Gamma_{T'}(\psi^{-1})\Big).$$

Using the Fundamental Orthogonality relation, the sum over ψ evaluates to [10]

$$M_{R,(r,s)\alpha\beta;T,(t,u)\gamma\delta} = -g_{YM}^2 \sum_{R'} \frac{c_{RR'}d_T nm}{d_{R'} d_t d_u (n+m)} \mathrm{Tr}\Big(\big[\Gamma_R((1,m+1)), P_{R\to(r,s)\alpha\beta}\big] I_{R'T'}\times$$
$$\times \big[\Gamma_T((1,m+1)), P_{T\to(t,u)\gamma\delta}\big] I_{T'R'}\Big).$$

This expression for the dilatation operator is exact in N. For us to obtain the spectrum of anomalous dimensions, we need to consider the action of the dilatation operator on normalised operators. Normalised operators which do have unit two-point function can be obtained from

$$\chi_{R,(r,s)\alpha\beta}(Z,Y) = \sqrt{\frac{\mathcal{F}_R\,\mathrm{hooks}_R}{\mathrm{hooks}_r\,\mathrm{hooks}_s}} O_{R,(r,s)\alpha\beta}(Z,Y).$$

The action of the dilatation operator on these normalised operators are given as [10]

$$DO_{R,(r,s)\alpha\beta}(Z,Y) = \sum_{T,(t,u)\gamma\delta} N_{R,(r,s)\alpha\beta;T,(t,u)\gamma\delta} O_{T,(t,u)\gamma\delta}(Z,Y),$$

where

$$N_{R,(r,s)\alpha\beta;T,(t,u)\gamma\delta} = -g_{YM}^2 \frac{c_{RR'}d_T nm}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{\mathcal{F}_T\,\mathrm{hooks}_T\,\mathrm{hooks}_r\,\mathrm{hooks}_s}{\mathcal{F}_R\,\mathrm{hooks}_R\,\mathrm{hooks}_t\,\mathrm{hooks}_u}}\times$$
$$\times\ \mathrm{Tr}\Big(\big[\Gamma_R((1,m+1)), P_{R\to(r,s)\alpha\beta}\big] I_{R'T'} \big[\Gamma_T((1,m+1)), P_{T\to(t,u)\gamma\delta}\big] I_{T'R'}\Big).$$

There are three objects which appear in the expression above: the symmetric group operators (projectors) $P_{R\to(r,s)\alpha\beta}$, the intertwiners $I_{T'R'}$ and the symmetric group element $\Gamma_R((1,m+1))$. The projection operator has already been discussed in great detail, so the next two subsections will discuss $I_{T'R'}$ and $\Gamma_R((1,m+1))$.

2.6.1 Intertwiners

In this section we will consider the sum over S_{n+m-1}. When S_n acts on $V^{\otimes n}$ for $n > 1$, it provides a reducible representation. Imagine that this includes the irreps R and S. Representing the action of σ as matrix $\Gamma(\sigma)$, we can write

$$\Gamma(\sigma) = \begin{bmatrix} \Gamma_R(\sigma) & 0 & \dots \\ 0 & \Gamma_S(\sigma) & \dots \\ \dots & \dots & \dots \end{bmatrix}.$$

If we restrict ourselves to an S_{n-1} subgroup of S_n, then in general, both R and S will subduce a number of representations. Suppose that R subduces R'_1 and R'_2 and that S subduces S'_1 and S'_2. Then, for $\sigma \in S_{n-1}$ we have

$$\Gamma(\sigma) = \begin{bmatrix} \Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \dots \\ 0 & \Gamma_{R'_2}(\sigma) & 0 & 0 & \dots \\ 0 & 0 & \Gamma_{S'_1} & 0 & \dots \\ 0 & 0 & 0 & \Gamma_{S'_2} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

Imagine that as Young diagrams $S'_1 = R'_1$, in other words, one of the irreps subduced by R is isomorphic to one of the representations subduced by S. Then, a simple application of the Fundamental Orthogonality relations gives us

$$\begin{aligned}
&\sum_{\sigma \in S_{n-1}} \begin{bmatrix} \Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{ij} \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \Gamma_{S'_1}(\sigma) & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{ab} \\
&= \frac{(n-1)!}{d_{R'_1}} \delta_{R'_1 S'_1} \begin{bmatrix} 0 & 0 & \mathbf{1} & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{ib} \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \mathbf{1} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{aj} \\
&\equiv \frac{(n-1)!}{d_{R'_1}} \delta_{R'_1 S'_1} (I_{R'_1 S'_1})_{ib} (I_{S'_1 R'_1})_{aj},
\end{aligned}$$

where the form of the intertwiners has been given. Intertwiners are maps between two isomorphic spaces. For $\sigma \in S_{n-1}$

$$I_{R'T'} \Gamma_{T'}(\sigma) = \Gamma_{R'}(\sigma) I_{R'T'}.$$

We see that intertwiners share a similar operation as intertwining maps discussed in section 2.5.1, where intertwining maps where used to define restricted characters. The box removed to obtain R' and T' can be removed from any corner of the Young diagram. Since the first box is removed from R or T the intertwiner acts on the first slot of $V_p^{\otimes m}$. The delta function $\delta^{i_1}_{i_{\psi(1)}}$ freezes the 1 index and hence the S_{n+m-1} subgroup of S_{n+m} is obtained by keeping all elements on the first slot of S_{n+m} that leave the index 1 unchanged. Due to the choice that the intertwiner acts on the first slot of $V_p^{\otimes m}$, we see that the first slot corresponds to index i_1. Thus, the explicit form of the intertwiner is determined once the location of the box removed from T and the box removed from R are specified. For example, for the Young diagrams shown below, we have

$$I_{R'T'} = E_{1,5} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \quad I_{T'R'} = E_{5,1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1},$$

where E_{ij} (for $I_{R'T'}$) is the operator that removes a box in row i in Young diagram R to get R', and removes a box in row j in Young diagram T to get T'. Figure 3 shows R and the box that

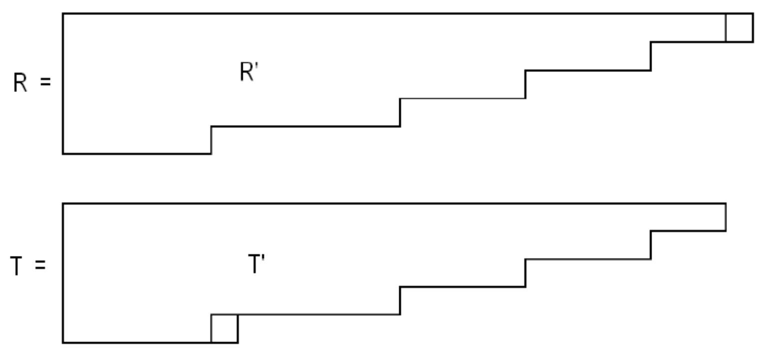


Figure 3: This figure illustrates R and the box that must be removed to obtain R', and T and the box that must be removed to obtain T'. As Young diagrams, $T' = R'$. Also, T and R both have 5 rows.

must be removed in order to obtain R' and T and the box that must be removed in order to obtain T'. As Young diagrams, $T' = R'$ and both T and R have 5 rows. We want to get the general result from this example. Let us first consider the case that $R \neq T$. To obtain R' from R, we remove a box from row i and to obtain T' from T, we remove a box from row j. Thus, in this situation, we have

$$I_{R'T'} = E_{ij} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \quad I_{T'R'} = E_{ji} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}.$$

In the case that $R = T$, the box that must be removed can be removed from any row and we get a contribution to the dilatation operator from each possible removal. Each possible removal must be represented by a different intertwiner and one needs to sum over all possible intertwiners. In this situation, the possible intertwiners are

$$I_{R'T'} = E_{kk} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} = I_{T'R'}, \qquad k = 1, 2, \ldots, p.$$

2.6.2 Symmetric Group Element

This group element acts on one slot from the Ys and one slot from the Zs. The box removed from R to get R' is the box acted on by the intertwiner and it is a Y box. This is one of the boxes that $\Gamma_R(1, m+1)$ acts on. The second box that $\Gamma_R(1, m+1)$ acts on can be any box associated to the Zs. Projectors and intertwiners only have an action on the boxes corresponding to Ys and as a result, the discussion has always taken place in the vector space $V_p^{\otimes m}$. However, because $\Gamma_R(1, m+1)$ acts on a Z box we must include one more slot and work in $V_p^{\otimes m+1}$. The intertwiners and projectors have a trivial action on the $(m+1)^{\text{th}}$ slot and hence the $(m+1)^{\text{th}}$ slot is simply occupied with the identity. Acting in $V_p^{\otimes m+1}$, $\Gamma_R(1,m+1)$ has a very simple action: it simply swaps the 1st and the $(m+1)^{\rm th}$ slots. The projectors when acting on $V_p^{\otimes m+1}$ are given by

$$\mathcal{P}_{R\to(r,s)\alpha\beta} = p_{R\to(r,s)\alpha\beta}\otimes\mathbf{1},$$

where the $p\times p$ unit matrix $\mathbf{1}$ acts on the $(m+1)^{\rm th}$ slot, and $p_{R\to(r,s)\alpha\beta}$ acts only in $V_p^{\otimes m}$. For comparison, the projectors appearing in the definition of the restricted Schur polynomial are [1],[2]

$$P_{R\to(r,s)\alpha\beta} = p_{R\to(r,s)\alpha\beta}\otimes\mathbf{I}_r,$$

where $\mathbf{I}_r$ is the identity matrix acting on the carrier space of the S_n irrep r. We will make use of the formula

$$\mathbf{1} = \sum_{k=1}^{p} E_{kk}.$$

In evaluating the dilatation operator, we will need to take products of the intertwiners and $\Gamma_R(1,m+1)$. These products are evaluated below:

$$\begin{aligned}\Gamma_R(1,m+1)E_{ij}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1} &= \Gamma_R(1,m+1)\sum_{k=1}^{p}E_{ij}\otimes\mathbf{1}\otimes\cdots\otimes E_{kk}\\ &= \sum_{k=1}^{p}E_{kj}\otimes\mathbf{1}\otimes\cdots\otimes E_{ik}\end{aligned}$$

$$\begin{aligned}E_{ij}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}\Gamma_R(1,m+1) &= \sum_{k=1}^{p}E_{ij}\otimes\mathbf{1}\otimes\cdots\otimes E_{kk}\Gamma_R(1,m+1)\\ &= \sum_{k=1}^{p}E_{ik}\otimes\mathbf{1}\otimes\cdots\otimes E_{kj}\end{aligned}$$

$$\Gamma_R(1,m+1)E_{ij}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}\Gamma_R(1,m+1) = \mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes E_{ij}.$$

From now on we will write the E_{ij} with a superscript, indicating which slot E_{ij} acts on. In this notation we have

$$E_{ik}\otimes\mathbf{1}\otimes\cdots\otimes E_{kj} = E_{ik}^{(1)}E_{kj}^{(m+1)}.$$

2.6.3 Dilatation Operator Coefficient

We will now explain how to evaluate the value of the coefficient

$$g_{YM}^2\frac{c_{RR'}d_T nm}{d'_R d_t d_u(n+m)}\sqrt{\frac{\mathcal{F}_T\,\mathrm{hooks}_T\,\mathrm{hooks}_r\,\mathrm{hooks}_s}{\mathcal{F}_R\,\mathrm{hooks}_R\,\mathrm{hooks}_t\,\mathrm{hooks}_u}}$$

in the large N limit [10]. The Young diagrams R, T, r, t, s and u each have p-rows. We use the symbols R_i, T_i, r_i, t_i, s_i and u_i where $i=1,2,\dots,p$ to denote the number of boxes in each row respectively. We assume p is fixed to be $O(1)$. The top row (which is also the longest row) is the value $i=1$ and the bottom row (shortest row) has $i=p$. It is straight forward to argue that the product of hook lengths, in r for example, is

$$\text{hooks}_R = \frac{\prod_{i=1}^{p}(r_i+p-i)!}{\prod_{j<k}(r_j-r_k+k-j)}.$$

For the diagrams R and T, the row lengths R_i are of order N. Further, R and T differ by at most the placement of a single box. This implies that $R_i=T_i$ for all except two values of i, say $i=a,b$. For these values of i, we have

$$R_b=T_b+1, \qquad R_a=T_a-1.$$

This implies that

$$\frac{\text{hooks}_R}{\text{hooks}_T} = \frac{(T_a-1+p-a)!(T_b+1+p-b)!}{(T_a+p-a)!(T_b+p-b)!}\prod_{k\neq a,b}\frac{|T_a-T_k|+|k-a|}{|T_a-1-T_k|+|k-a|}\times$$
$$\times\prod_{k\neq a,b}\frac{|T_b-T_k|+|k-b|}{|T_b+1-T_k|+|k-b|}\frac{|T_b-T_a|+|a-b|}{|T_a-T_b-2|+|a-b|} = \frac{R_b}{R_a}(1+O(N^{-1})).$$

We will use R_+ to denote the row length of the row in R that is longer than the corresponding row in T and let R_- denote the row length of the row in R that is shorter than the corresponding row in T. Using this notation

$$\frac{\text{hooks}_R}{\text{hooks}_T} = \frac{R_+}{R_-}(1+O(N^{-1})).$$

A generalisation can be made for the other hook factors $\frac{\text{hooks}_r}{\text{hooks}_t}$ and $\frac{\text{hooks}_s}{\text{hooks}_u}$. Now, consider a Young diagram R' that is obtained by removing a single box from Young diagram R. Assuming this box is removed from row a, we have the following relation between the lengths of the rows in R and the lengths of the row in R':

$$R_i=R'_i, \quad i\neq a, \qquad R_a=R'_a+1.$$

Thus, we find

$$\frac{\text{hooks}_R}{\text{hooks}_{R'}} = \frac{(R_a+p-a)!}{(R_a+p-1-a)!}\prod_{j\neq a}\frac{|R_j-R_a-1|+|a-j|}{|R_j-R_a|+|a-j|} = R_a(1+O(N^{-1})).$$

The coefficient mentioned at the beginning of this section is multiplied by the trace over an (r,s) subspace. This trace produces a number of order 1 multiplied by $d_{r'}d_s$. The product of the coefficient and the trace now reduces to quantities that we have studied. Now all the ingredients needed to estimate the large N values of the combinations of symmetric group dimensions and hook factors that appear in the dilatation operator. Notice that both the product of the hook lengths and the dimensions of symmetric group irreps are invariant under the flips of the Young diagram which exchanges columns and rows. Thus, these conclusions can immediately be recycled when studying the case of p long columns. Next, recalling that $\mathcal{F}_R$ is the product of factors in Young diagram R and $R'=T'$, we learn that

$$c_{RR'}\sqrt{\frac{\mathcal{F}_T}{\mathcal{F}_R}} = \sqrt{c_{RR'}c_{TT'}},$$

where $c_{RR'}$ is the factor associated to the box that must be removed from R to obtain R' and $c_{TT'}$ is the factor associated to the box that must be removed from T to obtain T'.

2.6.4 Evaluating Traces

We now evaluate the trace [10]

$$\mathcal{T} = \mathrm{Tr}\left(\left[\Gamma_R((1,m+1)), P_{R\to(r,s)\alpha\beta}\right] I_{R'T'} \left[\Gamma_T((1,m+1)), P_{T\to(t,u)\gamma\delta}\right] I_{T'R'}\right).$$

We start by writing this trace as a sum of traces over $m+1$ slots (all the Y slots plus one Z slot) times a trace over $n-1$ slots (the remaining Z slots). The trace over the $n-1$ slots is over the carrier space R^{m+1} which is described by a Young diagram that can be obtained by removing $m+1$ boxes from R, or equivalently by removing one box from r or equivalently by removing one box from t – these all give the same Young diagram describing R^{m+1}. R^{m+1} has different shapes depending on where the $(m+1)^{\text{th}}$ box is removed. The results from the last subsection clearly imply that the dimension of symmetric group representation R^{m+1}, denoted $d_{R^{m+1}}$, depends on the details of this shape. If the $(m+1)^{\text{th}}$ box is removed from row i, denote this dimension by $d^i_{R^{m+1}}$. Our general strategy is then to trace over the last Z slot (the $(m+1)^{\text{th}}$ slot) which then leaves a trace over $V_p^{\otimes m}$. This trace is then evaluated using elementary $U(p)$ representation theory.

The box removed from R to obtain R' is removed from the b^{th} row of R and the box removed from T to obtain T' is removed from the a^{th} row of T. After tracing over the $n-1$ Z slots associated to R^{m+1} (this produces a factor of $d^b_{R^{m+1}}$), multiplying the symmetric group elements $(1,m+1)$ with the intertwiners and then tracing over the $(m+1)^{\text{th}}$ slot we obtain [10]

$$\mathcal{T} = -\delta_{ab}\delta_{RT}\delta_{(r,s)(t,u)}\delta_{\alpha\delta}\delta_{\beta\gamma}d^b_{R^{m+1}}\left[\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)\beta\gamma}E^{(1)}_{bb}\right) + \mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)\alpha\delta}E^{(1)}_{bb}\right)\right] +$$
$$+d^b_{R^{m+1}}\,\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)\gamma\beta}E^{(1)}_{bb}P_{T\to(t,u)\gamma\delta}E^{(1)}_{aa}\right) + d^b_{R^{m+1}}\,\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)\gamma\beta}E^{(1)}_{aa}P_{T\to(t,u)\gamma\delta}E^{(1)}_{bb}\right).$$

We now need to evaluate the traces over $V_p^{\otimes m}$. We will write the projector as

$$p_{R\to(r,s)\alpha\beta} = \sum_{a=1}^{d_s} |M_s^\alpha, a\rangle\langle M_s^\beta, a|,$$

where M_s^α and M_s^β label states from $U(p)$ irrep s which have the same Δ weight. The indices α and β range from $1,\dots,I(\Delta(M))$. The index a is a multiplicity index that, as a consequence of Schur-Weyl duality, is organised by representation s of the symmetric group S_m. To evaluate the traces over $V_p^{\otimes m}$ we need to allow $E^{(1)}_{kk}$ to act on the state $|M_s^\alpha, a\rangle$. The state $|M_s^\alpha, a\rangle$ was obtained by taking a tensor product of m copies (one for each slot) of the fundamental representation of $U(p)$. It is possible and useful to rewrite this state as a linear combination of states which are each the tensor product of the fundamental representation for the first slot with a state obtained by taking the tensor product of states of the remaining $m-1$ slots. This is a useful thing to do because then $E^{(1)}_{kk}$ has a particularly simple action on each state in the linear combination. Towards this end we can write (in the following 0 stands for a string of $p-1$ 0s)

$$|M_s^\alpha, a\rangle = \sum_{M_{s'}, M_{\mathbf{10}}} C^{M_s^\alpha}_{M_{s'}, M_{\mathbf{10}}} |M_{\mathbf{10}}\rangle \otimes |M_{s'}, b\rangle,$$

where $M_{\mathbf{10}}$ indexes a state in the carrier space of the fundamental representation and $C^{M_s^\alpha}_{M_{s'}, M_{\mathbf{10}}}$ are Clebsch-Gordon coefficients given as

$$C^{M_s^\alpha}_{M_{s'}, M_{\mathbf{10}}} = \left(\langle M_{\mathbf{10}}| \otimes \langle M_{s'}, b|\right)|M_s^\alpha, a\rangle.$$

The representation s' is obtained by removing a single box from s. By appealing to the Schur-Weyl duality which organises the space $V_p^{\otimes m-1}$, we know that the multiplicity index b of the state $|M_{s'}, b\rangle$ is organised by the irrep s' of S_{m-1}. This allows us to easily evaluate the action of $E_{kk}^{(1)}$: it simply projects onto the state corresponding to box 1 sitting in the k^{th} row. Evaluating the traces over $V_p^{\otimes m}$ is now straight forward.

2.6.5 Long Columns

The computation of the action of the dilatation operator for restricted Schur polynomials labelled by Young diagrams that have a total of p long rows [1],[2] has made extensive use of the fact that we can organise the space of partially labelled Young diagrams into $S_n \times S_m$ irreps (r, s) by appealing to the Schur-Weyl duality. We have already argued that it is also possible to perform this organisation when considering restricted Schur polynomials labelled by Young diagrams that have a total of p long columns – all that is required is that we fine tune a few phrases in our map between partially labelled Young diagrams and vectors in $V_p^{\otimes m}$ [1],[2]. The same irreps of $U(p)$ are used for both these organisations, and further since $d_s = d_{s^\intercal}$, each $U(p)$ representation s appears with the same multiplicity in these two cases. Consequently, the traces computed in the last subsection for labels with p long rows are equal to the values for labels with p long columns. To obtain the action of the dilatation operator, all that remains is the computation of the coefficient. The only quantity appearing in the subsection "**Dilatation Operator Coefficient**" which is not invariant under exchanging rows and columns is

$$c_{RR'}\sqrt{\frac{\mathcal{F}_T}{\mathcal{F}_R}} = \sqrt{c_{RR'}c_{TT'}}.$$

This factor is the only difference between the case of p long rows and p long columns. Consequently, the action of the dilatation operator on restricted Schur polynomials with p long columns is obtained from its action on restricted Schur polynomials with p long rows by making substitutions of the form $N + b \to N - b$ [1],[2]. This completes the evaluation of the action of the dilatation operator.

2.7 Gauss graphs

Operators of good scaling dimension, that are dual to Gauss configuration σ [11], are

$$O_{R,r}(\sigma) = \frac{|H|}{\sqrt{m!}} \sum_{j,k} \sum_{s\vdash m} \sum_{\mu_1,\mu_2} \sqrt{d_s}\Gamma_{jk}^{(s)}(\sigma) B_{j\mu_1}^{s\to 1_H} B_{k\mu_2}^{s\to 1_H} O_{R,(r,s)\mu_1\mu_2},$$

where H is a subgroup of group G, j, k are labels for matrix representation Γ, s labels Young diagram with m boxes, and $B_{i\mu_1}^{s\to 1_H}$ [11] give the expansion of the μ^{th} occurrence of the identity of H when irrep s of S_m is decomposed into irreps of subgroup H, in terms of states labelled i in s. The two point function of Gauss graph operators is

$$\begin{aligned}\langle O_{R,r}(\sigma_1) O_{T,t}^\dagger(\sigma_2)\rangle &= \frac{|H|^2}{m!} \sum_{s,u\vdash m} \sum_{\mu_1\mu_2\nu_1\nu_2} \sqrt{d_s d_u}\Gamma_{jk}^{(s)}(\sigma_1)) B_{j\mu_1}^{s\to 1_H} B_{k\mu_2}^{s\to 1_H} \times \\ &\quad \times \Gamma_{lm}^{(u)}(\sigma_2)) B_{l\nu_1}^{u\to 1_H} B_{m\mu_2}^{u\to 1_H} \langle O_{R,(r,s)\mu_1\mu_2} O_{T,(t,u)\nu_1\nu_2}^\dagger\rangle \\ &= \sum_{\gamma_1,\gamma_2\in H} \delta(\sigma_1^{-1}\gamma_1\sigma_2\gamma_2).\end{aligned}$$

The last line comes about with the use of

$$\langle O_{R,(r,s)\mu_1\mu_2} O^\dagger_{T,(t,u)\nu_1\nu_2} \rangle = \delta_{rt}\delta_{su}\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}$$

and detailed manipulations found in section 3 in [11].

Young diagram R that is studied has p rows each with a length of $O(N)$ number of boxes and the difference in the number boxes compared between neighbouring rows is of $O(N)$ as well. The number of rows in Young diagram R corresponds to the number of dots seen in the Gauss graph illustration in figure 4 (which represent giant graviton branes); thus there are p dots. These Gauss graphs obey the Gauss Law constraint that states that the total charge on the giant's world volume must vanish. This is illustrated by the number of incoming and outgoing open strings is the same resulting in a zero overall charge for each dot (brane).

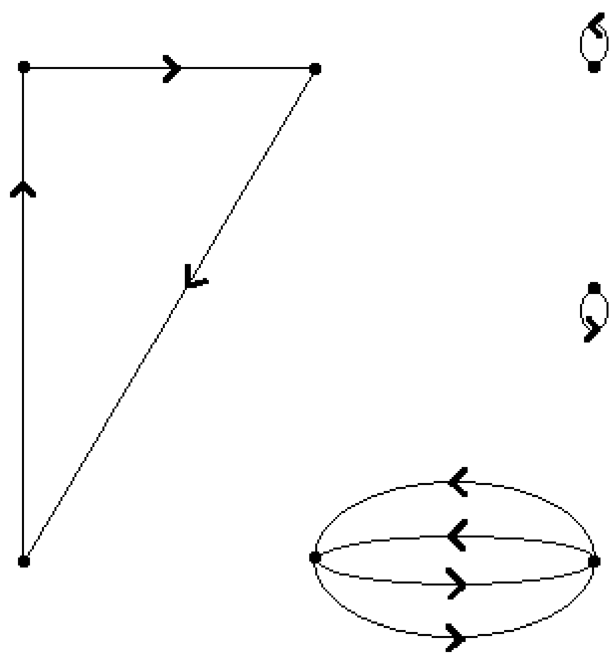


Figure 4: In this example we have 7 dots where for each dot the number of open strings leaving the dot and going to the dot is the same. Thus, these Gauss graphs obey the Gauss Law.

The action of the dilatation operator on the Gauss graph operator is [11]

$$DO_{R,r}(\sigma) = -g^2_{\mathrm{YM}} \sum_{i<j} \Delta_{ij} O_{R,r}(\sigma),$$

where g_{YM} is the coupling of the Yang-Mills theory, the parameter $n_{ij}(\sigma)$ counts the number of strings stretching from node i to node j. The operator Δ_{ij} when acted on a Young diagram R (or Young diagram r) results in the product of the sum of the content of the box at the end of the i^{th} row and the content of the box at the end of the j^{th} row (i.e. $c_i + c_j$) with the Young diagram R (or r) along with the product of the negative square root of the product of contents c_i and c_j (i.e. $-\sqrt{c_i c_j}$) with the Young diagram R (or r) where the box at the end of the i^{th} row is moved to the end of the j^{th} row and with the Young diagram R (or r) where the box at the end of the j^{th} row is moved to the end of the i^{th} row.

2.8 Conclusion

We have reviewed compelling evidence that suggests that matrix models are string theories. Amongst the evidence, it was argued that factorisation is connected to the conjecture that the large N limit of $\mathcal{N} = 4$ SYM theory is given by the classical limit of IIB string theory on $AdS_5 \times S^5$, with $1/N^2$ is equal to $\hbar_{\text{string theory}}$. When we computed correlators in the planar limit, we made use of ribbons graphs. In the large N but non-planar limit, we need to make use of Group Representation Theory. We study Gauss graphs since they are dual to operators of good scaling dimension and we study the action of the dilatation operator on these Gauss graph operators since we learn of the "energy" eigenvalues of the theory due to the fact that the dilatation operator is our Hamiltonian. All of that we have covered in this section provides the motivation to our question: is there an emergent Yang-Mills theory coming from the low energy description of branes and open strings that provides a description of operators that are dual to giant gravitons?

3 Spherical Harmonics on the 3-sphere

Our goal in this research project is to demonstrate how an emergent Yang-Mills theory provides an effective description of operators that are dual to giant gravitons. An emergent Yang-Mills theory is the expected world volume dynamics of giant gravitons. Since the world volume of a giant graviton is a 3-sphere, we expect the operators dual to giant gravitons have a description in terms of an effective Yang-Mills theory on a 3-sphere, S^3. To explore the spectrum of the theory, we want to perform a "Fourier expansion" on S^3; this is the subject of harmonic analysis. The techniques used to perform such expansions are developed using group theory. In this chapter, these ideas are developed to obtain the expansion of $\mathcal{N} = 4$ SYM on S^3. We will start by reviewing the Fundamental Orthogonality relation, obtained by appealing to Schur's Lemmas. Next, we will discuss how to perform a Fourier transformation on finite groups, which will then lead to the discussion of the Fourier (and an inverse Fourier) transformation on a coset. After that we construct the spherical harmonics of the coset $SO(4)/SO(3)$ which is the 3-sphere, S^3. Once we have the spherical harmonics, we carry out two checks to verify the validity of these harmonics.

3.1 The Fundamental Orthogonality Relation

The development and discussion of the Fundamental Orthogonality relation is imperative for Fourier analysis on groups and cosets. We will start off with the idea of an irreducible matrix representation of a group, and use concepts from group theory and Schur's two lemmas, to reach an understanding of what the Fundamental Orthogonality relation is. The Fundamental Orthogonality relation implies that the matrix elements of the irreducible representations are a complete set of functions on the group.

Consider the collection of matrices given as

$$[B(R,S,b,\alpha)]_{a\beta} = \sum_{g\in\mathcal{G}} \left[\Gamma_R(g^{-1})\right]_{ab} [\Gamma_S(g)]_{\alpha\beta}\,.$$

Notice that

$$\begin{aligned}
[B(R,S,b,\alpha)]_{a\beta} [\Gamma_S(g_1)]_{\beta\gamma} &= \sum_{g\in\mathcal{G}} \left[\Gamma_R(g^{-1})\right]_{ab} [\Gamma_S(g)]_{\alpha\beta} [\Gamma_S(g_1)]_{\beta\gamma} \\
&= \sum_{g\in\mathcal{G}} \left[\Gamma_R(g^{-1})\right]_{ab} [\Gamma_S(g)\Gamma_S(g_1)]_{\alpha\gamma} \\
&= \sum_{g\in\mathcal{G}} \left[\Gamma_R(g^{-1})\right]_{ab} [\Gamma_S(gg_1)]_{\alpha\gamma}\,.
\end{aligned}$$

We now change the summation variables from g to $\bar{g} = gg_1$ and note that $g = \bar{g}g_1^{-1} \Rightarrow g^{-1} = g_1\bar{g}^{-1}$. The above expression becomes

$$\begin{aligned}
[B(R,S,b,\alpha)]_{a\beta} [\Gamma_S(g_1)]_{\beta\gamma} &= \sum_{\bar{g}\in\mathcal{G}} \left[\Gamma_R(g_1\bar{g}^{-1})\right]_{ab} [\Gamma_S(\bar{g})]_{\alpha\gamma} \\
&= \sum_{\bar{g}\in\mathcal{G}} [\Gamma_R(g_1)]_{ac} \left[\Gamma_R(\bar{g}^{-1})\right]_{cb} [\Gamma_S(\bar{g})]_{\alpha\gamma}\,.
\end{aligned}$$

The sum over g was a sum over $|\mathcal{G}|$ distinct terms, that each belong to $\mathcal{G}$, so that we were summing over the set $\mathcal{G}$. Now, the $\bar{g}$'s we sum over belong to the group. Further, note that if

$$g_a \neq g_b \Rightarrow g_a g_1 \neq g_b g_1 \Rightarrow \bar{g}_a \neq \bar{g}_b,$$

which implies that we sum $\bar{g}$ over $|\mathcal{G}|$ distinct elements, and thus we conclude that the range of summation for $\bar{g}$ is also $\mathcal{G}$. The expression for the collection of matrices now becomes

$$\begin{aligned}[B(R,S,b,\alpha)]_{a\beta}\,[\Gamma_S(g_1)]_{\beta\gamma} &= [\Gamma_R(g_1)]_{ac}\sum_{\bar{g}\in\mathcal{G}}\Gamma_R(\bar{g}^{-1})_{cb}\,[\Gamma_S(\bar{g})]_{\alpha\gamma} \\ &= [\Gamma_R(g_1)]_{ac}\,[B(R,S,b,\alpha)]_{c\gamma}\,.\end{aligned}$$

Since $B\Gamma_S(g_1) = \Gamma_R(g_1)B$ for all $g_1 \in \mathcal{G}$, by Schur's second Lemma (see Appendix A.1.2 for the full description), B is proportional to a delta function of the irreps R and S, i.e.

$$[B(R,S,b,\alpha)]_{c\gamma} \propto \delta_{RS},$$

and by Schur's first Lemma (see Appendix A.1.1 for the full description)

$$[B(R,S,b,\alpha)]_{c\gamma} = \delta_{RS}\delta_{c\gamma}\lambda(b,\alpha,R),$$

where $\lambda(b,\alpha,R)$ is still to be determined. By taking the trace of this result, we find that

$$\begin{aligned}\mathrm{Tr}\,(B(R,R,b,\alpha)) &= \sum_{c=1}^{d_R}[B(R,R,b,\alpha)]_{cc} \\ &= d_R\lambda(b,\alpha,R) \\ &= \sum_{c}\sum_{g\in\mathcal{G}}\left[\Gamma_R(g^{-1})\right]_{cb}[\Gamma_R(g)]_{\alpha c} \\ &= \sum_{g\in\mathcal{G}}\left[\Gamma_R(g)\Gamma_R(g^{-1})\right]_{\alpha b} \\ &= \sum_{g\in\mathcal{G}}\left[\Gamma_R(gg^{-1})\right]_{\alpha b} \\ &= \sum_{g\in\mathcal{G}}\delta_{\alpha b} \\ &= |\mathcal{G}|\delta_{\alpha b}\end{aligned}$$

$$\therefore \lambda(b,\alpha,R) = \frac{|\mathcal{G}|}{d_R}\delta_{\alpha b}.$$

Thus the Fundamental Orthogonality relation is given as

$$[B(R,S,b,\alpha)]_{a\beta} = \sum_{g\in\mathcal{G}}\left[\Gamma_R(g^{-1})\right]_{ab}[\Gamma_S(g)]_{\alpha\beta} = \frac{|\mathcal{G}|}{d_R}\delta_{RS}\delta_{a\beta}\delta_{\alpha b}.$$

This relation clearly implies that the matrix elements of the irreducible representations are a complete set of orthogonal functions on the group.

3.2 Fourier transform on a finite group

In this section, motivated by the Fundamental Orthogonality relation, we will introduce the notion of a Fourier transform on a (finite) group. The Fourier transform is used to move from position space ($f(x)$) to momentum space ($f(p)$). On the group, the Fourier transform is used to move from a function on the group ($f(\sigma)$) to a function on the matrix elements of the irreducible representations ($f_{R,a,b}$ with $a, b = 1, \ldots, d_R$ with d_R the dimension of irreducible representation R). At the end of this section we will apply this logic to the group of translations which leads to the usual formulas for the Fourier transform.

Consider a real function on the group, for example

$$f : G \to \mathbb{R},$$

such that $f(\sigma)$ is equal to a real number, for each choice of $\sigma \in S_n$. The order of the group is

$$|G| = n! = \text{number of distinct } \sigma.$$

The vector space in which $f(\sigma)$ lives is thus $\mathbb{R}^{n!}$. A basis for this space is provided by the matrix elements of the irreducible representations $\Gamma_R(\sigma)_{ab}$ which can be used to define, a vector, i.e.

$$\Gamma_R(\sigma)_{ab} \equiv |R, a, b\rangle_\sigma,$$

where R, a, b is the "name" of the vector; σ indexes the components of the vector. Every representation of S_n is equivalent to an orthogonal representation. Thus, we can assume, without loss of generality, that we have an orthogonal representation. In this case

$$\Gamma_S(\sigma^{-1})_{ab} = \Gamma_S(\sigma)_{ba} = |S, b, a\rangle_\sigma,$$

The collection of matrices obey the Fundamental Orthogonality relation, so that

$$\sum_{\sigma \in S_n} {}_\sigma\langle S, d, c|R, a, b\rangle_\sigma = \sum_{\sigma \in S_n} \Gamma_R(\sigma)_{ab}\Gamma_S(\sigma^{-1})_{cd} = \frac{|G|}{d_R}\delta_{RS}\delta_{ad}\delta_{bc},$$

which tells you that the vectors are orthogonal to each other (this is the result of the Fundamental Orthogonality Relation.) We thus conclude that

$$\Gamma_R(\sigma)_{ab} = n! \text{ linearly independent vectors which constitute a basis for } \mathbb{R}^{n!}.$$

Expanding $f(\sigma)$ in this basis we find that, there is a Fourier transform that takes us from $f_{R,a,b}$ to $f(\sigma)$ and it is given as

$$f(\sigma) = \sum_{R,a,b} f_{R,a,b}\, \Gamma_R(\sigma)_{ab}. \tag{2}$$

The inverse Fourier transform is given as

$$f_{R,a,b} = \frac{d_R}{|G|}\sum_{\sigma \in S_n} f(\sigma)\Gamma_R(\sigma^{-1})_{ba}. \tag{3}$$

Using the Fundamental Orthogonality relation, it is simple to show that (2) and (3) are indeed consistent.

We can think of these transformations as a Fourier and inverse Fourier transformation as follows: we can think of the function $f(\sigma)$ as the "position space" function. The Fundamental Orthogonality relation implies that the matrix elements of the irreducible representations of the group are orthogonal functions on the group (they are a complete set of functions). Thus we have the Fourier transform described in (2) and the inverse Fourier transform described in (3) where $f_{R,a,b}$ can be thought of as the "momentum space" function.

To make the connection even more explicit, consider the group of translations, with group composition law

$$T(x_1)T(x_2) = T(x_1 + x_2), \quad x_1, x_2 \in \mathbb{R}.$$

In this case, a function on the group actually is a function in position space. Since the group of translations is an Abelian group, all of its irreducible representations are one dimensional. They are given by

$$T_k(x) = e^{ik\cdot x}, \quad k \in \mathbb{R},$$

where k is a label for the irreducible representation. Recall that the Fourier transform we identified above is given by

$$f(\sigma) = \sum_{R,a,b} f_{R,a,b}\, \Gamma_R(\sigma)_{ab}.$$

To specialise this formula to the example we are considering, note the following:

1. Since all irreducible representations are one dimensional, we don't need indices a,b. Denoting group elements by x (instead of σ) and labelling the irreducible representations with k (instead of R) we replace

$$f(\sigma) \to f(x), \quad f_{R,a,b} \to \tilde{f}(k), \quad \Gamma_R(\sigma)_{ab} \to T_k(x) = e^{ik\cdot x}.$$

2. Since the irreducible representation of the group of translations are now labelled by a continuous real number, we replace the sum over R with an integral over k. The Fourier transform thus becomes

$$f(x) = \int_{-\infty}^{\infty} dk\ \tilde{f}(k) e^{ik\cdot x}$$

which is indeed just the usual Fourier transform. It is also instructive to note that the Fundamental Orthogonality relation becomes

$$\begin{aligned}\int_{-\infty}^{\infty} dx\ T_k(x)T_{k'}(-x) &= \int_{-\infty}^{\infty} dx\ e^{ik\cdot x - ik'\cdot x} \\ &= 2\pi\delta(k - k').\end{aligned}$$

Return to the discussion for the finite group. Recall that the Fourier transform pair relevant for this case is given by

$$\phi^R_{ij} = \sum_{g\in G} \Gamma^R_{ij}(g^{-1})\phi(g), \tag{4}$$

$$\phi(g) = \sum_{R,i,j} \frac{d_R}{|G|} \Gamma^R_{ij}(g)\phi^R_{ij}. \tag{5}$$

These two equations are equations (3) and (2) respectively; the reason the equations are repeated is due to a change in normalisation and use of alphabet letters. We can plug (5) into the right hand side of (4) and test if the two sides are equal. To carry this out we need to use following expression for the delta function on the group

$$\delta(g) = \sum_R \frac{d_R}{|G|}\chi_R(g).$$

$\delta(g)$ is 1 if g is the identity; otherwise it vanishes. The derivation of this delta function can be found in appendix C.

We wish to explicitly demonstrate that the Fourier transform pair is valid. Towards this end, plug (5) into (4), to get

$$\begin{aligned}
\phi^R_{ij} &= \sum_{g\in G} \Gamma^R_{ij}(g^{-1}) \left[\sum_{S,k,l} \frac{d_S}{|G|}\Gamma^S_{kl}(g)\phi^S_{lk}\right] \\
&= \sum_{g\in G}\sum_{S,k,l} \frac{d_S}{|G|}\Gamma^R_{ij}(g^{-1})\Gamma^S_{kl}(g)\phi^S_{lk} \\
&= \sum_{S,k,l} \frac{d_S}{|G|}\left[\sum_{g\in G}\Gamma^R_{ij}(g^{-1})\Gamma^S_{kl}(g)\right]\phi^S_{lk} \\
&= \sum_{S,k,l} \frac{d_S}{|G|}\frac{|G|}{d_R}\delta_{RS}\delta_{il}\delta_{jk}\phi^S_{lk} \\
&= \phi^R_{ij}.
\end{aligned}$$

In the third to fourth line, we made use of the Fundamental Orthogonality Relation. When we plug (4) into (5), we get

$$\begin{aligned}
\phi(g) &= \sum_{R,i,j} \frac{d_R}{|G|}\Gamma^R_{ij}(g)\left[\sum_{\tilde{g}\in G}\Gamma^R_{ji}(\tilde{g}^{-1})\phi(\tilde{g})\right] \\
&= \sum_{R,i,j}\sum_{\tilde{g}\in G} \frac{d_R}{|G|}\Gamma^R_{ij}(g)\Gamma^R_{ji}(\tilde{g}^{-1})\phi(\tilde{g}) \\
&= \sum_{R}\sum_{\tilde{g}\in G} \frac{d_R}{|G|}\left(\sum_{i,j}\Gamma^R_{ij}(g)\Gamma^R_{ji}(\tilde{g}^{-1})\right)\phi(\tilde{g}) \\
&= \sum_{R}\sum_{\tilde{g}\in G} \frac{d_R}{|G|}\chi_R(g\tilde{g}^{-1})\phi(\tilde{g}) \\
&= \sum_{\tilde{g}\in G}\left[\sum_R \frac{d_R}{|G|}\chi_R(g\tilde{g}^{-1})\right]\phi(\tilde{g}) \\
&= \sum_{\tilde{g}\in G}\delta(g\tilde{g}^{-1})\phi(\tilde{g}) \\
&= \phi(g).
\end{aligned}$$

In the fifth to sixth line, we used the delta function on the group. We have thus demonstrated that (4) and (5) are consistent and are a valid Fourier transform pair.

To properly identify the Fourier transform on the coset, we need to generalize these four equations: we need a Fourier transform and an inverse transform, as well as two delta functions.

3.3 Coset Manifolds

The 3-sphere is an example of a manifold that can be written as a coset. Thus to understand harmonic analysis on the 3-sphere, we need to understand harmonic analysis on a coset. In this section we will explain how the sphere is described using the coset language. The reason we discuss cosets is because we need to "divide out" the subgroup H from the group G, to eliminate group operations in such a way that we obtain a unique identification between group operations and points in a manifold. To make this clear, let's work with the coset $SO(3)/SO(2)$, where the $SO(3)$ group describes rotations in three dimensions and the $SO(2)$ group in this case describes the rotation about the z-axis. We want to go from one point to another on the surface of the

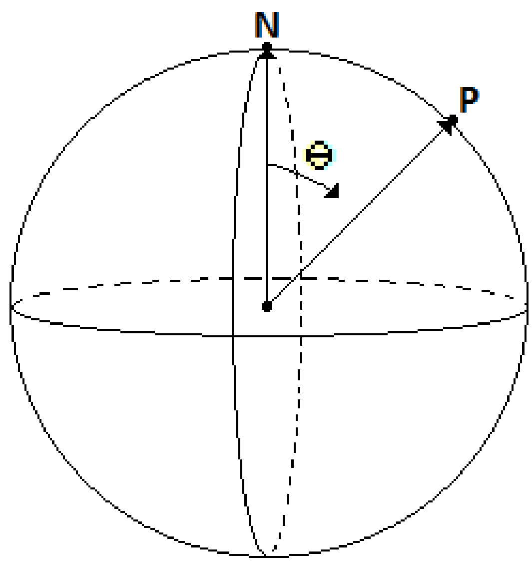


Figure 5: We have a sphere where N, the north pole, is the point at which the vector at the origin initially points to, and the point P which is the desired point on the surface of the sphere that we rotate to once the operation of θ has been performed.

sphere, starting at the north pole, N, of the sphere. We can simply go there by rotating through some angle θ, which will take us to any desired point P. This operation allows us to identify any point on the surface of the sphere with a group element of the group $SO(3)$. This identification of points on the sphere and elements of $SO(3)$ only becomes unique once we impose an equivalence relation, $g \cdot h \sim g$, where $g \in SO(3)$, $h \in SO(2)$. This is because we could rotate first about the z-axis and then rotate by the angle θ or we could simply have rotated by the angle θ. The result is the same. What is clear is that there is more than one way to go from the North pole to another point on the surface of the sphere; this is because the rotation about the z-axis did not change the position you were at before the rotation so we must get rid of the operation of rotating about the z-axis to obtain a unique association between elements of $SO(3)$ and points in S^2. By "dividing out" the group $SO(2)$, we can thus identify elements of the coset $SO(3)/SO(2)$ with points on the

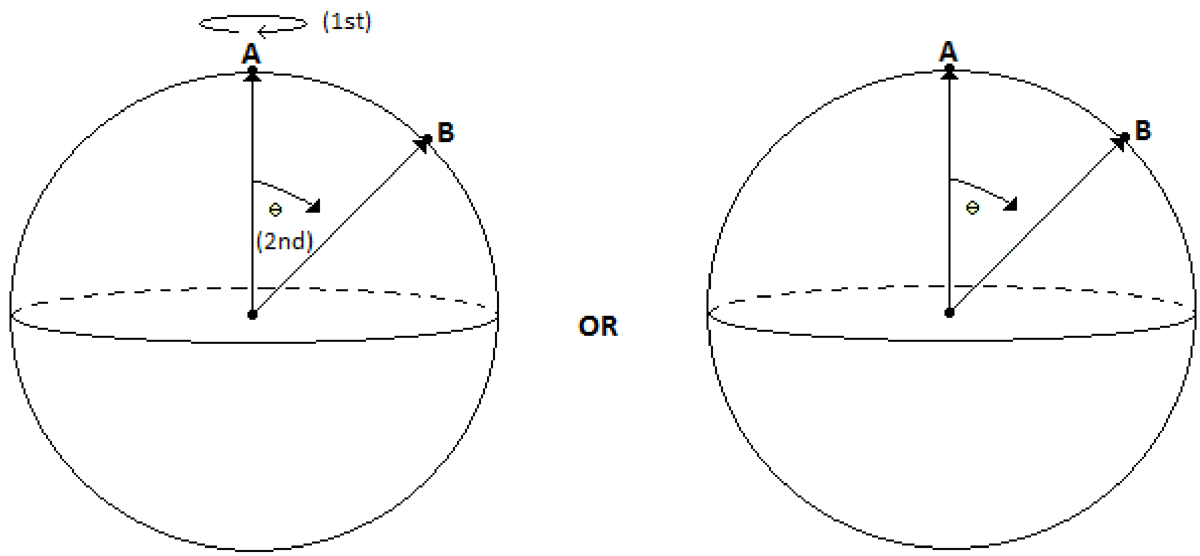


Figure 6: The first diagram demonstrates that one can, initially at point **A**, rotate about the z-axis and then perform the operation of θ to get to point **B**. The second diagram demonstrates that one can simply only perform the operation of θ to get to point **B**. The operation of rotating about the z-axis plays an unimportant role for the case considered.

surface of the 2-sphere, S^2. Through a similar argument, the coset $SO(4)/SO(3)$ is the 3-sphere, S^3.

The coset picture can provide us with co-ordinates for the manifold and provides a powerful approach to the differential geometry of the manifold. We will illustrate this by deriving the metric on S^2, using the coset description.

Considering the rotations about the x-axis

$$R_{23}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix},$$

y-axis

$$R_{31}(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

and z-axis

$$R_{12}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using the standard relation $R_{ij}(\theta) = e^{i\theta T_{ij}}$, we can determine the generators of infinitesimal rotations, T_{ij}. To do this, rotate by the infinitesimal angle ε and expand R_{ij} to first order, i.e. $R_{ij}(\varepsilon) = e^{i\varepsilon T_{ij}} = \mathbb{1} + i\varepsilon T_{ij}$, and then solve for T_{ij}. To illustrate the procedure, consider the computation of T_{12}

$$
\begin{aligned}
R_{12}(\varepsilon) &= \begin{bmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + i\varepsilon T_{12} \\
\Rightarrow i\varepsilon T_{12} &= \begin{bmatrix} 0 & -\varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = i\varepsilon \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\therefore T_{12} &= \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -i \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

T_{31} and T_{23} are determined in a similar fashion and they are

$$
T_{31} = -i \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}
$$

and

$$
T_{23} = -i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.
$$

Consider the rotation, $L = R_{12}(\phi)R_{31}(\theta)$, and allow it to act on the Cartesian basis vector $\hat{e}_z$ to obtain

$$
\begin{aligned}
R_{12}(\phi)R_{31}(\theta)\hat{e}_z &= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \cos\phi\sin\theta \\ \cos\theta\sin\phi & \cos\phi & \sin\phi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{bmatrix}. \qquad (6)
\end{aligned}
$$

The vector above correctly gives the Cartesian co-ordinates for each point on the S^2

$$
\begin{aligned}
x &= \cos\phi\sin\theta \\
y &= \sin\phi\sin\theta \\
z &= \cos\theta.
\end{aligned}
$$

This explicitly shows how each point is identified with an element $R_{12}(\phi)R_{31}(\theta) \in SO(3)/SO(2)$. To compute the metric of the 2-sphere, we consider the differential

$$
\begin{aligned}
ds = L^{-1}dL &= \left(e^{i\phi T_{12}}e^{i\theta T_{31}}\right)^{-1} d\left(e^{i\phi T_{12}}e^{i\theta T_{31}}\right) \\
&= e^{-i\theta T_{31}}e^{-i\phi T_{12}}\left[id\phi\ T_{12}\ e^{i\phi T_{12}}e^{i\theta T_{31}} + e^{i\phi T_{12}}\ id\theta\ T_{31}\ e^{i\theta T_{31}}\right] \\
&= id\phi\ e^{-i\theta T_{31}}e^{-i\phi T_{12}}\ T_{12}\ e^{i\phi T_{12}}e^{i\theta T_{31}} + id\theta\ e^{-i\theta T_{31}}\ T_{31}\ e^{i\theta T_{31}} \\
&= id\phi\ R_{31}(-\theta)R_{12}(-\phi)T_{12}R_{12}(\phi)R_{31}(\theta) + id\theta\ R_{31}(-\theta)T_{31}R_{31}(\theta).
\end{aligned}
$$

Note that $R_{12}(-\phi)T_{12}R_{12}(\phi) = T_{12}$ and $R_{31}(-\theta)T_{31}R_{31}(\theta) = T_{31}$ since the rotations and counter-rotations about their respective generators leaves the generators invariant. Thus the differential becomes

$$
\begin{aligned}
ds &= id\phi \; R_{31}(-\theta)T_{12}R_{31}(\theta) + id\theta \; T_{31} \\
&= id\phi \begin{bmatrix} 0 & i\cos\theta & 0 \\ -i\cos\theta & 0 & -i\sin\theta \\ 0 & i\sin\theta & 0 \end{bmatrix} + id\theta \; T_{31} \\
&= id\phi \left[\cos\theta \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \right] + id\theta \; T_{31} \\
&= id\phi\cos\theta T_{12} + id\phi\sin\theta(-T_{23}) + id\theta \; T_{31} \\
&= \cos\theta d\phi(iT_{12}) + \sin\theta d\phi(-iT_{23}) + d\theta(iT_{31}) \\
&\equiv \cos\theta d\phi \; \hat{n}_{12} + \sin\theta d\phi \; \hat{n}_{23} + d\theta \; \hat{n}_{31}.
\end{aligned}
$$

Note, however, this is the differential for the group $SO(3)$. Since we are working with the coset $SO(3)/SO(2)$, we divide out the term involving $\hat{n}_{12}$, so that our differential is

$$ds = \sin\theta d\phi \; \hat{n}_{23} + d\theta \; \hat{n}_{31}.$$

The terms $\hat{n}_{23}$ and $\hat{n}_{31}$ correspond to small displacements in the tangent space and hence contribute to ds. The term $\hat{n}_{12}$ can be used to construct the spin connection [12]. This leads to

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

which is indeed the metric for the 2-sphere of unit radius.

3.4 Fourier transform on a coset

Having established the relevance of cosets for the geometry of the spheres S^2 and S^3, in this section we will develop the Fourier transform on a coset.

Consider a function $f(\sigma)$ on the coset, G/H, where each coset element is an equivalence class of G, i.e.

$$\sigma_1 \sim \sigma_2 \iff \sigma_1 = \sigma_2 h \; , \; \sigma_1, \sigma_2 \in G \; , \; h \in H.$$

Note that the order of the coset G/H is smaller than the order of the group G. In any representation R we can write

$$\Gamma_R(\sigma_1)_{AB} = \Gamma_R(\sigma_2)_{AC}\Gamma_R(h)_{CB},$$

where representation R reduces as $R = r_1 \oplus r_2 \oplus \cdots \oplus r_s$ when we restrict G to H. We can use the original indices A,B to refer to matrix elements of $\Gamma_R(\sigma)$. Alternatively we can use indices a,b that refer to matrix elements of the irreducible representation of H. Since any particular irreducible representation of H may appear more than once, we need a label γ telling us which copy of a given irrep of H appears, as well as a label a that refers to the matrix elements of the irrep. We can freely translate between A and a,γ. Thus, we could write matrix elements as $\Gamma_R(\sigma_1)_{AB}$, $\Gamma_R(\sigma_1)_{A,a\gamma}$, or even $\Gamma_R(\sigma_1)_{b\beta,a\gamma}$.

3.4.1 Cosets of finite groups

We are considering the coset G/H. We use upper case letters from the end of the alphabet to discuss irreps of G and use upper case letters from the beginning of the alphabet to discuss their matrix elements. We use lower case letters from the end of the alphabet to discuss irreps of H and use lower case letters from the beginning of the alphabet to discuss their matrix elements. To spell out the notation, a formula that we are familiar with is

$$\Gamma^R_{aA}(hg) = \Gamma^r_{ab}(h)\Gamma^R_{bA}(g) \qquad g \in G, \quad h \in H.$$

This notation should be improved: when we restrict from G to H, r may appear more than once when we restrict R. Thus, we need to have a multiplicity index that goes with the matrix elements of r - these tell us which copy of r we are considering. We will use a Greek letter to denote this multiplicity. With this improved notation, the above equation becomes

$$\Gamma^R_{a\alpha,A}(hg) = \Gamma^r_{ab}(h)\Gamma^R_{b\alpha,A}(g).$$

Functions on the coset give a vector valued function in some carrier space of an irreducible representation of H

$$\phi_a(g) = \phi_a(h\tilde{g}) = \Gamma^r_{ab}(h)\phi_b(\tilde{g}).$$

The guess for the Fourier and Inverse Fourier transform are now

$$\phi_a(g) = \sum_R \sum_{A=1}^{d_R} \sum_\alpha \Gamma^R_{a\alpha,A}(g)\phi^R_{A\alpha}, \tag{7}$$

$$\phi^R_{A,\alpha} = \frac{d_R|H|}{d_r|G|} \sum_{L\in G/H} \sum_{a=1}^{d_r} \Gamma^R_{A,a\alpha}(L)\phi_a(L^{-1}). \tag{8}$$

We also need two delta functions. We will start from the Fundamental Orthogonality Relation. We can trade the index A for the pair of indices $a\alpha$. After doing that, the Fundamental Orthogonality Relation becomes

$$\sum_{g\in G} \Gamma^R_{A,a\alpha}(g)\Gamma^S_{b\beta,B}(g^{-1}) = \frac{|G|}{d_R}\delta_{RS}\delta_{AB}\delta_{\alpha\beta}\delta_{ab}.$$

Expressing $g = Lh$ with $L \in G/H$ and $h \in H$ we can write the sum over G as a sum over H and a sum over G/H as follows

$$\sum_{h\in H} \sum_{L\in G/H} \Gamma^R_{A,c\alpha}(L)\Gamma^r_{ca}(h)\Gamma^s_{bd}(h^{-1})\Gamma^S_{d\beta,B}(L^{-1}) = \frac{|G|}{d_R}\delta_{RS}\delta_{AB}\delta_{\alpha\beta}\delta_{ab}.$$

We can now perform the sum over H using the Fundamental Orthogonality Theorem

$$\frac{|H|}{d_r}\delta_{rs}\delta_{ab} \sum_{L\in G/H} \Gamma^R_{A,c\alpha}(L)\Gamma^S_{c\beta,B}(L^{-1}) = \frac{|G|}{d_R}\delta_{RS}\delta_{AB}\delta_{\alpha\beta}\delta_{ab}.$$

Rearranging a little, we have

$$\sum_{L\in G/H} \Gamma^R_{A,c\alpha}(L)\delta_{cd}\Gamma^S_{d\beta,B}(L^{-1}) = \frac{|G|}{|H|}\frac{d_r}{d_R}\delta_{RS}\delta_{AB}\delta_{\alpha\beta}\delta_{rs}.$$

The dependence on the irreps r, s on the left-hand side is a little hidden: c belongs to r and d to s. This is the first delta function we will need.

Now, to derive the second delta function we will need, consider

$$\sum_R \sum_\alpha \frac{d_R}{d_r} \frac{|H|}{|G|} \Gamma^R_{a\alpha,b\alpha}(gL^{-1}).$$

Using the projector which projects onto a matrix element of the irrep r we have[2]

$$O^{R,r}_{ab} \equiv \frac{d_r}{|H|} \sum_{h\in H} \Gamma^r_{ab}(h^{-1}) \Gamma^R_{BA}(h) = \sum_\alpha \delta_{A,a\alpha} \delta_{b\alpha,B}.$$

Review appendix D for the proof that $O^{R,r}_{ab}$ is a projector. Then

$$\begin{aligned}
\sum_R \sum_\alpha \frac{d_R}{d_r} \frac{|H|}{|G|} \Gamma^R_{a\alpha,b\alpha}(gL^{-1}) &= \sum_R \sum_{h\in H} \frac{d_R}{|G|} \Gamma^r_{ab}(h^{-1}) \chi_R(hgL^{-1}) \\
&= \sum_{h\in H} \Gamma^r_{ab}(h^{-1}) \sum_R \frac{d_R}{|G|} \chi_R(hgL^{-1}) \\
&= \sum_{h\in H} \Gamma^r_{ab}(h^{-1}) \delta(hgL^{-1}).
\end{aligned}$$

This is in fact a delta function on the coset! Indeed, imagine that $g = h_1 L$ so that g and L represent the same coset element. Then

$$\sum_R \sum_\alpha \frac{d_R}{d_r} \frac{|H|}{|G|} \Gamma^R_{a\alpha,b\alpha}(gL^{-1}) = \sum_{h\in H} \Gamma^r_{ab}(h^{-1}) \delta(hgL^{-1}) = \Gamma^r_{ab}(h_1)$$

Thus, this delta function is

$$\begin{aligned}
\sum_R \sum_\alpha \frac{d_R}{d_r} \frac{|H|}{|G|} \Gamma^R_{a\alpha,b\alpha}(gL^{-1}) &= 0 \qquad gL^{-1} \notin H \\
&= \Gamma^r_{ab}(h_1) \qquad gL^{-1} = h_1 \in H.
\end{aligned}$$

It will now be demonstrated that (7) and (8) as a Fourier transform pair are consistent. When we plug (8) into (7), we get

$$\begin{aligned}
\phi_a(g) &= \sum_R \sum_{A=1}^{d_R} \sum_\alpha \Gamma^R_{a\alpha,A}(g) \left[\frac{d_R |H|}{d_r |G|} \sum_{L\in G/H} \sum_{b=1}^{d_r} \Gamma^R_{A,b\alpha}(L) \phi_b(L^{-1}) \right] \\
&= \sum_R \sum_\alpha \sum_{L\in G/H} \sum_{b=1}^{d_r} \frac{d_R |H|}{d_r |G|} \left[\sum_{A=1}^{d_R} \Gamma^R_{a\alpha,A}(g) \Gamma^R_{A,b\alpha}(L) \right] \phi_b(L^{-1}) \\
&= \sum_R \sum_\alpha \sum_{L\in G/H} \sum_{b=1}^{d_r} \frac{d_R |H|}{d_r |G|} \Gamma^R_{a\alpha,b\alpha}(gL) \phi_b(L^{-1}) \\
&= \sum_{L\in G/H} \sum_{b=1}^{d_r} \left[\sum_R \sum_\alpha \frac{d_R |H|}{d_r |G|} \Gamma^R_{a\alpha,b\alpha}(gL) \right] \phi_b(L^{-1}).
\end{aligned}$$

[2] Calling this a projector is a slight abuse of notation. It is an intertwining map. We will continue this abuse.

Now, we are summing over the complete coset so that we can change variable from L to L^{-1}. We will denote $g = h_1\tilde{L}$. What we now have is

$$\begin{aligned}
\phi_a(g) &= \sum_{L\in G/H}\sum_{b=1}^{d_r}\left[\sum_R\sum_\alpha \frac{d_R|H|}{d_r|G|}\Gamma^R_{a\alpha,b\alpha}(gL^{-1})\right]\phi_b(L) \\
&= \sum_{L\in G/H}\sum_{b=1}^{d_r}\Gamma^r_{ab}(h_1)\phi_b(L)\delta_{L\tilde{L}} \\
&= \sum_{b=1}^{d_r}\Gamma^r_{ab}(h_1)\phi_b(\tilde{L}) \\
&= \phi_a(g).
\end{aligned}$$

When we plug (7) into (8), we get

$$\begin{aligned}
\phi^R_{A,\alpha} &= \frac{d_R|H|}{d_r|G|}\sum_{L\in G/H}\sum_{a=1}^{d_r}\Gamma^R_{A,a\alpha}(L)\left[\sum_S\sum_{B=1}^{d_S}\sum_\beta \Gamma^S_{a\beta,B}(L^{-1})\phi^S_{B,\beta}\right] \\
&= \frac{d_R|H|}{d_r|G|}\sum_{b=1}^{d_r}\sum_S\sum_{B=1}^{d_S}\sum_\beta\left[\sum_{L\in G/H}\Gamma^R_{A,b\alpha}(L)\delta_{ba}\Gamma^S_{a\beta,B}(L^{-1})\right]\phi^S_{B,\beta} \\
&= \frac{d_R|H|}{d_r|G|}\sum_{b=1}^{d_r}\sum_S\sum_{B=1}^{d_S}\sum_\beta\left(\frac{|G|d_r}{|H|d_R}\delta_{RS}\delta_{AB}\delta_{\alpha\beta}\delta_{rs}\right)\phi^S_{B,\beta} \\
&= \sum_S\sum_{B=1}^{d_S}\sum_\beta \delta_{RS}\delta_{AB}\delta_{\alpha\beta}\phi^S_{B,\beta} \\
&= \phi^R_{A,\alpha}.
\end{aligned}$$

Thus, it has been demonstrated explicitly that the Fourier transform pair are indeed consistent.

3.4.2 Lie groups

Having thoroughly developed the harmonic expansion for cosets of finite groups, the generalisation to cosets of Lie groups is straight forwards. Indeed, with a convenient choice for the normalisations involved, we have [12]

$$\phi_i(g) = \sum_n\sum_{\zeta,q}\sqrt{\frac{d_n}{d_{\mathbb{D}}}}D^n_{i\zeta,q}(g)\phi^n_{q\zeta},$$

where $\phi_i(g)$ is the set of fields labelled by i and are functions of $g \in G$. $D^n(g)$ is irreducible representation n of G while $\mathbb{D}(h)$ is an irreducible representation of H. The index i refers to matrix elements of $\mathbb{D}(h)$, while ζ tells us which copy of $\mathbb{D}(h)$ we are considering. Finally, q refers to matrix elements of $D^n(g)$. d_n and $d_{\mathbb{D}}$ are the dimensions of D and the particular representation of H, $\mathbb{D}(h)$, respectively. Thus, $q = 1, 2, \ldots, d_n$ and $i = 1, 2, \ldots, d_{\mathbb{D}}$. The coefficients $\phi^n_{q\zeta}$ are given by the inverse transform [12]

$$\phi^n_{q\zeta} = \frac{1}{V_k}\sqrt{\frac{d_n}{d_{\mathbb{D}}}}\int_{G/H} d\mu\ D^n_{q,l\zeta}(L_y)\phi_l(L_y^{-1}),$$

where V_k is the volume of G/H, $d\mu$ is the invariant measure on G/H that is normalised to the volume, and L_y is an element of G/H.

From the above expressions it is clear that the scalar, vector, tensor, ... harmonics used to perform the Fourier transform are given by the matrix elements $D^n_{i\zeta,q}(g)$. In the next section we will use this observation to explicitly construct the spherical harmonics needed to carry out the Fourier transform on the sphere.

3.5 Constructing spherical harmonics from cosets

The harmonics that will be used to carry out the expansion on the sphere are given by suitable matrix elements of the matrices representing the group. To make contact with the usual spherical harmonics we need representations of $SO(3)$. To construct these representations, we exponentiate the generators of the group, which are nothing but the angular momenta.

Work in the basis in which the z-component of the angular momentum operator is a diagonal matrix

$$L_z = \begin{bmatrix} l & 0 & \cdots & \cdots & 0 \\ 0 & l-1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \cdots & -(l-1) & 0 \\ 0 & 0 & \cdots & 0 & -l \end{bmatrix},$$

where $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. In this basis, the raising and lowering operators $L_\pm$ are matrices that have entries on the diagonal above the main diagonal and on the diagonal below the main diagonal respectively. The entries are given by

$$L_\pm |l, m\rangle = \sqrt{l(l+1) - m(m \pm 1)} |l, m \pm 1\rangle,$$

where $\hbar$ is set to 1, $|l, m\rangle$ is an angular momentum eigenket, and $-l \leq m \leq l$. We can compute the x- and y-component of the angular momentum operators as follows

$$L_x = \frac{L_+ + L_-}{2},$$
$$L_y = \frac{L_+ - L_-}{2i}.$$

When working with the coset $SO(3)/SO(2)$, which is the 2-sphere and $SO(2)$ is the rotation about the z-axis, we discover that when restricting to the $SO(2)$ subgroup, to compute $e^{i\phi L_z}$, we obtain a block diagonal matrix representation. One of the entries of $e^{i\phi L_z}$ is "1" and this is the subspace to which we need to project to obtain the scalar representation. To illustrate this, consider the spin-1 representation. The generators for this representation are

$$L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$L_+ = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$L_- = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}.$$

Then by restricting to the $SO(2)$ subgroup, we get

$$e^{i\phi L_z} = \begin{bmatrix} e^{i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\phi} \end{bmatrix},$$

which is a block diagonal matrix with 3 blocks. States in row 2 and column 2 have $L_z = 0$ (which leads to $e^{i\phi L_z}$ being equal to 1, which is the "1" that was being referred to before), and to get the scalar representation, we need to project to the 2 subspace. The parametrising of our coset is done as follows (this was discussed in equation (6))

$$D_{qq'}(\theta, \phi) = \left(e^{i\phi L_z} e^{i\theta L_y}\right)_{qq'}.$$

For us to get the spherical harmonics of the coset harmonic expansion $D^R_{q,i\zeta}$, it is required that we leave the row index alone and project the column index onto the scalar representation of the subgroup. This amounts to taking

$$Y_q = \sum_{q'} D_{qq'}(\theta, \phi) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{q'} = \sum_{q'} (e^{i\phi L_z} e^{i\theta L_y})_{qq'} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{q'}.$$

We obtain the following

$$Y_1 \propto e^{i\phi} \sin\theta,$$

$$Y_2 \propto \cos\theta,$$

$$Y_3 \propto e^{-i\phi} \sin\theta,$$

which are indeed the expected spherical harmonics. Now consider the coset $SO(4)/SO(3)$, which is the 3-sphere. $SO(3)$ is the group of rotations in the 12, 13 and 23 planes. The $SO(4)(= SU_L(2) \times SU_R(2))$ irrep is denoted by (j_L, j_R) where "L" and "R" mean "left" and "right" respectively. Once the irrep has been built, it gives us two sets of generators namely A_i and B_i, which obey the following algebra,

$$[A_i, A_j] = i\varepsilon_{ijk} A_k, \quad [B_i, B_j] = i\varepsilon_{ijk} B_k, \quad [A_i, B_j] = 0, \quad \forall i, j.$$

where $i, j, k = 1, 2, 3$. It is important to note that the spin j irrep for A_i,B_i is the set of $(2j+1) \times (2j+1)$ matrices that obey the algebra defined in the equation above. The relation between A_i and B_i and the 6 generators of the $SO(4)$ group: T_{12}, T_{23}, T_{13}, T_{14}, T_{24}, and T_{34}, is

$$\begin{aligned} T_{23} &= A_1 + B_1 \\ T_{13} &= A_2 + B_2 \\ T_{12} &= A_3 + B_3 \\ T_{14} &= A_1 - B_1 \\ T_{24} &= A_2 - B_2 \\ T_{34} &= A_3 - B_3. \end{aligned}$$

The coset can be parametrised as follows

$$D_{qq'}(\theta,\phi,\psi) = \left(e^{-i\theta T_{14}} e^{i\phi T_{12}} e^{i\psi T_{23}}\right)_{qq'},$$

where now the A_i's are the tensor product between the specific generators of the spin irrep chosen acting on the "left" space and the identity acting on the "right" space. The opposite is true for the B_i's, i.e., they are the tensor product between the identity of acting on the "left" space and the specific generators of the spin irrep chosen acting on the "right" space. The spin j irrep of $SO(4)$, after restricting to $SO(3)$, can be explicitly described as

$$(j_1, j_2) = j_1 \otimes j_2 = \sum_{j=|j_1-j_2|}^{j_1+j_2} j = |j_1 - j_2| \oplus (|j_1 - j_2| + 1) \oplus \cdots \oplus (j_1 + j_2 - 1) \oplus (j_1 + j_2).$$

We can relate the above equation to arrangements of boxes to create Young diagrams, through the tensor product of boxes, where a single box represents a particle with spin-$\frac{1}{2}$. For example, the tensor between two spin-$\frac{1}{2}$ particles is

$$\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \sum_{j=|\frac{1}{2}-\frac{1}{2}|}^{\frac{1}{2}+\frac{1}{2}} j = \left(\mathbf{\frac{1}{2}} - \mathbf{\frac{1}{2}}\right) \oplus \left(\mathbf{\frac{1}{2}} + \mathbf{\frac{1}{2}}\right) = \mathbf{0} \ \oplus \ \mathbf{1},$$

which is related to stacking the two boxes as follows

$$\square \otimes \square = \begin{array}{c}\square\\[-4pt]\square\end{array} \oplus \square\!\square,$$

where on the right hand side, the first diagram represents spin-0 and the second diagram represents spin-1. When we consider the dot product between two arbitrary, 3-component based vectors, $\vec{u}$ and $\vec{v}$, we naturally get $\vec{u}\cdot\vec{v} = u^1v^1 + u^2v^2 + u^3v^3$, which can also be rewritten as

$$\begin{aligned}\vec{u}\cdot\vec{v} = u^1v^1 + u^2v^2 + u^3v^3 &= \frac{(u^1 + iu^2)}{\sqrt{2}}\frac{(v^1 - iv^2)}{\sqrt{2}} + \frac{(u^1 - iu^2)}{\sqrt{2}}\frac{(v^1 + iv^2)}{\sqrt{2}} + u^3v^3 \\ &\equiv u^+v^- + u^-v^+ + u^3v^3,\end{aligned} \tag{9}$$

where

$$u^+ \equiv \frac{u^1 + iu^2}{\sqrt{2}}, \ u^- \equiv \frac{u^1 - iu^2}{\sqrt{2}}, \ v^+ \equiv \frac{v^1 + iv^2}{\sqrt{2}}, \ v^- \equiv \frac{v^1 - iv^2}{\sqrt{2}}.$$

In this case, we treat the "vector" of the form $e^{i\phi L_z^{(1)}\otimes\mathbb{1}}$ as our vector $\vec{u}$ in the sense that the diagonal entries act as the components of $\vec{u}$. To elaborate more on this, entries we expect to get from $e^{i\phi L_z^{(1)}\otimes\mathbb{1}}$ are proportional to $e^{i\phi}$, $e^{-i\phi}$, and $e^{i\phi\times 0} = 1$. These components are related to u^+, u^-, and u^3 respectively. In the same way, we treat the "vector" of the form $e^{\mathbb{1}\otimes i\phi L_z^{(2)}}$ as our vector $\vec{v}$ where now the diagonal entries act as the components of $\vec{v}$. As in the case for $e^{i\phi L_z^{(1)}\otimes\mathbb{1}}$, entries we expect to get from $e^{\mathbb{1}\otimes i\phi L_z^{(2)}}$ are also proportional to $e^{i\phi}$, $e^{-i\phi}$, and $e^{i\phi\times 0} = 1$, but are in a different configuration to $e^{i\phi L_z^{(1)}\otimes\mathbb{1}}$. Now, these components are related to v^+, v^-, and v^3 respectively. The dot product is an invariant quantity and we wish to construct the invariant quantity between the vectors $e^{i\phi L_z^{(1)}\otimes\mathbb{1}}$ and $e^{\mathbb{1}\otimes i\phi L_z^{(2)}}$. To do that, we inspect the diagonal elements of each vector and name them out in the order in which they appear. We multiply the components of the two vectors together in the precise order in which they appear in the diagonal. To illustrate this, suppose our two vectors have the following form:

$$e^{i\phi L_z^{(1)}\otimes\mathbb{1}} = \begin{bmatrix} e^{i\phi} &&&&&&&& \\ & 1 &&&&&&& \\ && e^{-i\phi} &&&&&& \\ &&& e^{i\phi} &&&&& \\ &&&& 1 &&&& \\ &&&&& e^{-i\phi} &&& \\ &&&&&& e^{i\phi} && \\ &&&&&&& 1 & \\ &&&&&&&& e^{-i\phi} \end{bmatrix},$$

$$e^{\mathbb{1}\otimes i\phi L_z^{(2)}} = \begin{bmatrix} e^{i\phi} &&&&&&&& \\ & e^{i\phi} &&&&&&& \\ && e^{i\phi} &&&&&& \\ &&& 1 &&&&& \\ &&&& 1 &&&& \\ &&&&& 1 &&& \\ &&&&&& e^{-i\phi} && \\ &&&&&&& e^{-i\phi} & \\ &&&&&&&& e^{-i\phi} \end{bmatrix},$$

then the dot product between $\vec{u}$ and $\vec{v}$ is

$$\vec{u}\cdot\vec{v} = u^+v^+ + u^3v^+ + u^-v^+ + u^+v^3 + u^3v^3 + u^-v^3 + u^+v^- + u^3v^- + u^-v^-, \tag{10}$$

and thus the invariant quantity defined in equation (9) is obtained by projecting onto (see equation (10))

$$\text{invariant vector} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

We project our coset parametrisation onto the invariant vector to obtain the scalar representation of the subgroup, $Y_q(\theta,\phi,\psi)$. From this result, we can read off the spherical harmonics of the 3-sphere. We are now ready to obtain the scalar spherical harmonics of the irreps we choose. When looking at the $(\frac{1}{2},\frac{1}{2})$ irrep, our generators A_i and B_i are the generators of the $SU(2)$ group, namely $J_i = \frac{1}{2}\sigma_i$ where σ_i are the Pauli matrices. The exponentiated z-component generators are computed to be

$$e^{i\phi L_z^{(1)}\otimes\mathbb{1}} = \begin{bmatrix} e^{i\phi} & 0 & 0 & 0 \\ 0 & e^{i\phi} & 0 & 0 \\ 0 & 0 & e^{-i\phi} & 0 \\ 0 & 0 & 0 & e^{-i\phi} \end{bmatrix}$$

$$e^{\mathbb{1}\otimes i\phi L_z^{(2)}} = \begin{bmatrix} e^{i\phi} & 0 & 0 & 0 \\ 0 & e^{-i\phi} & 0 & 0 \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & 0 & 0 & e^{-i\phi} \end{bmatrix}$$

which leads to the dot product of the $\vec{u}$ and $\vec{v}$

$$\vec{u}\cdot\vec{v} = u^+v^+ + u^+v^- + u^-v^+ + u^-v^-,$$

which reflects the alignments that are possible with two spins: $\uparrow\uparrow$, $\uparrow\downarrow$, $\downarrow\uparrow$, and $\downarrow\downarrow$. Thus, the invariant quantity we need to project onto to get the scalar representation is the singlet. Therefore, our spherical harmonics vector is

$$Y_i(\theta,\phi,\psi) = \frac{1}{\sqrt{2}}(D_{i2} - D_{i3}).$$

By following the same procedure when we work with the $(1,1)$ irrep, we determine the spherical harmonics vector to be

$$Y_i(\theta,\phi,\psi) = D_{i3} - D_{i5} + D_{i7},$$

where the phase factors are precisely what the Clebsch-Gordan analysis gives. By projecting onto the singlet of $SO(3)$ we can produce all of the scalar spherical harmonics. By projecting onto the vector of $SO(3)$ we can produce the vector spherical harmonics.

The invariant quantity for the $(\frac{1}{2},\frac{1}{2})$ irrep needed to project to the vector representation is the triplet. Therefore, for the vector harmonics, we find

$$Y_i(\theta,\phi,\psi) = \begin{cases} D_{i1}, & \uparrow\uparrow \\ \frac{1}{\sqrt{2}}(D_{i2}+D_{i3}), & \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ D_{i4}, & \downarrow\downarrow\,. \end{cases}$$

3.6 Orthogonality and eigenvalue checks

We have two ways of checking the validity of the spherical harmonics we have obtained through the method described in the previous section: an orthogonality check and an eigenvalue check. The orthogonality check is explicitly described as follows:

$$\mathcal{N}\int_{S^3} Y_i(\theta,\phi,\psi)Y_j(\theta,\phi,\psi)d\Omega = \delta_{ij},$$

given that the right normalisation, $\mathcal{N}$, has been determined for $Y_i(\theta,\phi,\psi)$. We can explicitly write out the integration over the 3-sphere, S^3, as follows

$$\int_{S^3}(\dots)d\Omega = \int_0^{2\pi} d\psi \int_0^{\pi} d\phi \int_0^{\pi} d\theta\ r^3\sin^2\theta\sin\phi(\dots),$$

where if we work with a unit 3-sphere then $r = 1$. The eigenvalue check is done by operating on $Y_i(\theta, \phi, \psi)$ with the Laplacian of S^3 and it is described as follows

$$\nabla \cdot \nabla_{S^3} Y_i(\theta, \phi, \psi) = \nabla^2_{S^3} Y_i(\theta, \phi, \psi) = -\frac{2j(2j+2)}{R^2} Y_i(\theta, \phi, \psi),$$

where R is the radius of the 3-sphere. We work with a unit 3-sphere so that $R = 1$. j is the irrep we are working with, for example $(\frac{1}{2}, \frac{1}{2})$ where $j = \frac{1}{2}$ or $(1, 1)$ where $j = 1$. We can explicitly write out the operation of the 3-sphere Laplacian on a function $f = f(\theta, \phi, \psi)$ as follows

$$\nabla^2_{S^3} f = \frac{1}{\sin^2\theta \sin\phi} \left[\frac{\partial}{\partial\psi} \left(\frac{1}{\sin\phi} \frac{\partial f}{\partial\psi} \right) \right] + \frac{\partial}{\partial\phi} \left(\sin\phi \, \frac{\partial f}{\partial\phi} \right) + \frac{\partial}{\partial\theta} \left(\sin^2\theta \sin\phi \frac{\partial f}{\partial\theta} \right).$$

The irreps we chose to consider are $(\frac{1}{2}, \frac{1}{2})$ and $(1, 1)$. We find the following results:

<u>Orthogonality check (unit 3-sphere):</u>

- For $\left(\frac{1}{2}, \frac{1}{2}\right)$:

$$\int_{S^3} Y_i(\theta, \phi, \psi) d\Omega = \pi^2, \quad \forall i,$$

  thus

$$\mathcal{N} \int_{S^3} Y_i(\theta, \phi, \psi) d\Omega = 1, \; \mathcal{N} = \frac{1}{\pi} \quad \forall i.$$

- For $(1, 1)$:

$$\int_{S^3} Y_i(\theta, \phi, \psi) d\Omega = \frac{2\pi^2}{3}, \quad \forall i,$$

  thus

$$\mathcal{N} \int_{S^3} Y_i(\theta, \phi, \psi) d\Omega = 1, \; \mathcal{N} = \sqrt{\frac{3}{2} \frac{1}{\pi}} \quad \forall i.$$

<u>Eigenvalue check (unit 3-sphere):</u>

- For $\left(\frac{1}{2}, \frac{1}{2}\right)$:

$$\nabla^2_{S^3} Y_i(\theta, \phi, \psi) = -3 Y_i(\theta, \phi, \psi), \quad \forall i.$$

- For $(1, 1)$:

$$\nabla^2_{S^3} Y_i(\theta, \phi, \psi) = -8 Y_i(\theta, \phi, \psi), \quad \forall i.$$

These are convincing checks of the spherical harmonics for irreps $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $(1, 1)$.

A different way of checking both the eigenvalues of the scalar *and* vector spherical harmonics has been developed [12]. When the covariant derivative acts on a vector, it first displaces the vector infinitesimally:

$$x^\mu \to x^\mu + \delta x^\mu,$$
$$\epsilon^\mu \partial_\mu v^\nu = \delta v^\nu,$$

and secondly it rotates the vector according to the point in spacetime the vector is at:

$$\delta v^\nu = \epsilon^\mu \Gamma^\nu_{\mu\alpha} v^\alpha.$$

Thus, what we see is that

Small displacement = small group transformation = acting with the generator.

From this logic, the operation of the covariant derivative acting on our irrep $D^n_{qa}(\theta, \phi)$ is

$$\nabla_\alpha D^n_{qa}(\theta,\phi) = D^n_{qq'}(\theta,\phi) D^n_{q'a}(-iT_\alpha).$$

This operation is quite important since it tells us that the operation of the covariant derivative on our irrep can also be achieved by simply acting on the harmonics $D^n_{qq'}(\theta,\phi)$, that have the labels q, q' which are labels of the group G, and the label n which labels irreps of G, with the matrix $D^n_{q'a}(-iT_\alpha)$ that has the label q' *and* a which are labels of the group G and H respectively. The element $-iT_\alpha$ comes about from how we parametrised the coset, i.e. $e^{-i\theta T}$. Note that

$$D^n_{q'q}(-iT_\alpha) = (-iT_\alpha)^n_{q'q},$$

where T_α is the matrix for the generator in irrep n.

From the description of the operation of the covariant derivative, the Laplacian is then given as

$$\nabla^2 D^n_{qa}(\theta,\phi) = -D^n_{qq'}(\theta,\phi) D^n_{q'a}(T_\alpha T_\alpha), \tag{11}$$

where $T_\alpha T_\alpha = (T_{14})^2 + (T_{24})^2 + (T_{34})^2$. When we move on the surface of the 3-sphere, we start at the identity and rotate, by θ and ϕ, very closely in the proximity of the identity. The rotations in the 12, 23, and 13 planes do not change the position we are at before the rotations thus the operation $T_\alpha T_\alpha$ does not include them. However, we can re-write the dot product of the generators as follows

$$\begin{aligned} T_\alpha T_\alpha =& (T_{14})^2 + (T_{24})^2 + (T_{34})^2 + (T_{12})^2 + (T_{23})^2 + (T_{13})^2 \\ & - \left((T_{12})^2 + (T_{23})^2 + (T_{13})^2\right) \\ =& C_2(G) - C_2(H), \end{aligned}$$

where $C_2(G)$ is the quadratic Casimir of G and $C_2(H)$ is the quadratic Casimir of H. For (l_1, l_2) of $SO(4)$, $C_2(G) = 2(l_1(l_1+1) + l_2(l_2+1))$ and for s of $SO(3)$, $C_2(H) = s(s+1)$. We will quickly review the argument demonstrating that $T^a T^a$ is a Casimir. Using the Lie algebra

$$[T^a, T^b] = i f^{abc} T^c,$$

we find

$$\begin{aligned} [T^a T^a, T^b] &= [T^a, T^b] T^a + T^a [T^a, T^b] \\ &= i f^{abc} [T^c T^a + T^a T^c] \\ &= 0, \end{aligned}$$

since f^{abc} is anti-symmetric and $T^cT^a + T^aT^c$ is symmetric. This result is true for any T^b. Following from this, we see that

$$[T^aT^a, e^{i\omega^b T^b}] = 0$$

is also true. From Schur's Lemma, this implies that

$$T^aT^a = \lambda \cdot \mathbb{1},$$

where λ is some number; in our case we consider it to be an eigenvalue.

Before we look at the eigenvalue checks, let us first have a closer look at our irrep $D^n_{q'a}(T_\alpha T_\alpha)$:

$$\begin{array}{c} q' \text{ is the state in irrep } n \text{ of } G = |n, q'\rangle \\ a \text{ is the state in irrep } \mathbb{D} \text{ of } H = |\mathbb{D}, a\rangle \end{array}$$

thus

$$\begin{aligned} D^n_{q'a}(T_\alpha T_\alpha) &= \langle n, q'|T_\alpha T_\alpha|\mathbb{D}, a\rangle \\ &= \langle n, q'|(C_2(G) - C_2(H))|\mathbb{D}, a\rangle \\ &= \big(\langle n, q'|C_2(G)\big) - \big(C_2(H))|\mathbb{D}, a\rangle\big) \\ &= \big(\langle n, q'|2(l_1(l_1+1) + l_2(l_2+1))\big) - \big(s(s+1)|\mathbb{D}, a\rangle\big) \\ &= \langle n, q'|\big[2(l_1(l_1+1) + l_2(l_2+1)) - s(s+1)\big]|\mathbb{D}, a\rangle \\ &= \langle n, q'|\big[2(l_1(l_1+1) + l_2(l_2+1)) - s(s+1)\big]|\mathbb{D}, a\rangle \\ &\equiv \big[2(l_1(l_1+1) + l_2(l_2+1)) - s(s+1)\big]\langle n, q'|\mathbb{D}, a\rangle. \end{aligned}$$

For the irrep $\left(\frac{1}{2}, \frac{1}{2}\right)$, the eigenvalue checks are as follows:

- Scalar spherical harmonic:

$$\begin{aligned} \nabla^2 D^{1/2}_{(0)a}(\theta, \phi) &= -\big[2(l_1(l_1+1) + l_2(l_2+1)) - s(s+1)\big]^n_{q'a} D^n_{(0)q'}(\theta, \phi) \\ &= -\big[2(1/2(1/2+1) + 1/2(1/2+1)) - 0(0+1)\big]^n_{q'a} D^n_{(0)q'}(\theta, \phi) \\ &= -3D^n_{(0)q'}(\theta, \phi). \end{aligned}$$

- Vector spherical harmonic:

1.

$$\begin{aligned} \nabla^2 D^{1/2}_{(1)a}(\theta, \phi) &= -\big[2(l_1(l_1+1) + l_2(l_2+1)) - s(s+1)\big]^n_{q'a} D^n_{(1)q'}(\theta, \phi) \\ &= -\big[2(1/2(1/2+1) + 1/2(1/2+1)) - 1(1+1)\big]^n_{q'a} D^n_{(1)q'}(\theta, \phi) \\ &= -D^n_{(1)q'}(\theta, \phi). \end{aligned}$$

2.

$$\begin{aligned} \nabla^2 D^{1/2}_{(0)a}(\theta, \phi) &= -\big[2(l_1(l_1+1) + l_2(l_2+1)) - s(s+1)\big]^n_{q'a} D^n_{(0)q'}(\theta, \phi) \\ &= -\big[2(1/2(1/2+1) + 1/2(1/2+1)) - 1(1+1)\big]^n_{q'a} D^n_{(0)q'}(\theta, \phi) \\ &= -D^n_{(0)q'}(\theta, \phi). \end{aligned}$$

3.

$$\begin{aligned}\nabla^2 D^{1/2}_{(-1)a}(\theta,\phi) &= -\big[2(l_1(l_1+1)+l_2(l_2+1)) - s(s+1)\big]^{n}_{q'a} D^{n}_{(-1)q'}(\theta,\phi)\\ &= -\big[2(1/2(1/2+1)+1/2(1/2+1)) - 1(1+1)\big]^{n}_{q'a} D^{n}_{(-1)q'}(\theta,\phi)\\ &= -D^{n}_{(-1)q'}(\theta,\phi).\end{aligned}$$

These are the correct results since the eigenvalues of the scalar and vector spherical harmonics are

$$2j_1(j_1+1)+2j_2(j_2+1) = -3, \quad j_1, j_2 \in SO(4)$$

and

$$2j_1(j_1+1)+2j_2(j_2+1) - s(s+1) = -1, \quad j_1, j_2 \in SO(4) \text{ and } s \in SO(3)$$

respectively. In this case, j_1,j_2, and s are equal to $\frac{1}{2}$ since we are considering the irrep $\left(\frac{1}{2},\frac{1}{2}\right)$. Notice that the scalar spherical harmonics eigenvalue is the same as when we performed the first check. This is a nice check of equation (11).

3.7 Harmonic expansion of N=4 SYM on the 3-sphere

Now that we have reviewed harmonic expansion on spheres, we can consider the expansion of $\mathcal{N}=4$ super Yang-Mills theory. As we have just reviewed, the harmonics carry an irreducible representation (m_L, m_R) of the isometry group $SO(4) = SU(2)_L \times SU(2)_R$ of the sphere. Further, since we are on the coset $SO(4)/SO(3)$, they are in representations of $SO(3)$. The mode expansions [13], in Coulomb gauge $\nabla_a A^a = 0$ are

$$\phi_i(x) = \sum_{k=0}^{\infty}\sum_{I=1}^{(k+1)^2} \phi_i^{kI}(t) Y^{kI}_{(0)}(\mathbf{x}), \tag{10a}$$

$$\lambda^A_\alpha(x) = \sum_{k=0}^{\infty}\sum_{I=1}^{(k+1)(k+2)}\sum_{\pm} \lambda^{A,kI\pm}(t) Y^{kI\pm}_{(1/2)\alpha}(\mathbf{x}), \tag{10b}$$

$$A_0(x) = \sum_{k=0}^{\infty}\sum_{I=1}^{(k+1)^2} \omega_{kI}(t) Y^{kI}_{(0)}(\mathbf{x}), \tag{10c}$$

$$A_a(x) = \sum_{k=0}^{\infty}\sum_{I=1}^{(k+1)(k+3)}\sum_{\pm} A^{kI\pm}(t) Y^{kI\pm}_{(1)a}(\mathbf{x}). \tag{10d}$$

It is the modes collected in (10a) that will be of most interest in this dissertation. The harmonics are denoted as $Y^{kI}_{(s)}$. The index k singles out an irrep of $SO(4)$, as explained in the table below. The index I runs over the states in this $SO(4)$ irrep. Finally, s specifies the representation of $SO(3)$ that the harmonic belongs to. The spinor spherical harmonics are commuting (two-dimensional) Weyl spinors.

The table below shows the harmonics on S^3 that appear in the expansion of spin-0, spin-1/2 and spin-1 fields [13].

Spin	Harmonic functions	Irreps	Masses
0	Scalar spherical harmonics: $Y^{kI}_{(0)}$	$(\mathbf{k+1},\mathbf{k+1})$	$(k+1)/R$
1/2	Spinor spherical harmonics: $Y^{kI+}_{(1/2)}$, $Y^{kI-}_{(1/2)}$	$(\mathbf{k+2},\mathbf{k+1})$, $(\mathbf{k+1},\mathbf{k+2})$	$(k+\frac{3}{2})/R$
1	Vector spherical harmonics: $Y^{kI+}_{(1)}$, $Y^{kI-}_{(1)}$	$(\mathbf{k+3},\mathbf{k+1})$, $(\mathbf{k+1},\mathbf{k+3})$	$(k+2)/R$

Substituting the harmonic expansions into the action [13]

$$S = \frac{2}{g_{YM}^2} \int d^4x \sqrt{|g|}\, \mathrm{Tr} \Bigg[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} D^\mu \phi_i D_\mu \phi_i - \frac{\mathcal{R}}{12} \phi_i^2 + \frac{1}{4} [\phi_i, \phi_j]^2 - 2i \lambda_A^\dagger \sigma^\mu D_\mu \lambda^A \\ + (\rho_i)^{AB} \lambda_A^\dagger i\sigma^2 [\phi_i, \lambda_B^*] - (\rho_i^\dagger)_{AB} (\lambda^A)^\top i\sigma^2 [\phi_i, \lambda^B] \Bigg]$$

and computing the integrals over the three-sphere, we obtain the quantum mechanics of an infinite number of fields. To obtain the mass spectrum of these excitations, the integration over the quadratic terms in the action given is required. A couple of properties of the spherical harmonics are required before we perform the integration; these properties being the orthonormality and eigenvalue property. We have seen similar orthogonality and eigenvalue checks earlier in this section but only for the scalar spherical harmonics; we will now state the properties for the scalar and other spherical harmonics. The spherical harmonics are orthonormalised [13] to

$$\int_{S^3} Y_{(0)}^{kI} Y_{(0)}^{lJ} = \delta^{kl}\delta^{IJ}, \int_{S^3} \left(Y_{(1/2)\alpha}^{kI\pm}\right)^* Y_{(1/2)\alpha}^{kI\pm} = \delta^{kl}\delta^{IJ}, \int_{S^3} \left(Y_{(1)a}^{kI\pm}\right)^* Y_{(1)\alpha}^{kI\pm\ a} = \delta^{kl}\delta^{IJ}$$

and their eigenvalues of the Laplace-Beltrami operators [13] are

$$\begin{aligned} \nabla^2 Y_{(0)}^{kI} &= -\frac{1}{R^2} k(k+2) Y_{(0)}^{kI}, \\ \sigma^a \nabla_a &= \pm \frac{i}{R} \left(k + \frac{3}{2}\right) Y_{(1/2)}^{kI\pm}, \\ \nabla^2 Y_{(1/2)\alpha}^{kI\pm} &= -\frac{1}{R^2} \left[k(k+3) + \frac{3}{4}\right] Y_{(1/2)\alpha}^{kI\pm}, \\ \nabla^2 Y_{(1)a}^{kI\pm} &= -\frac{1}{R^2} \left[k(k+4) + 2\right] Y_{(1)a}^{kI\pm}. \end{aligned}$$

After substituting the mode expansion, (10a) to (10d) into the quadratic terms of the action [13], we obtain

$$\begin{aligned} S_{\text{quad}} = \frac{4\pi^2 R^3}{g_{YM}^2} \int dt \Bigg[& \sum_{k,I} \left(\frac{1}{2} \mathrm{Tr}\, \dot\phi_i^{kI} \dot\phi_i^{kI} - \frac{(k+1)^2}{2R^2} \mathrm{Tr}\, \phi_i^{kI} \phi_i^{kI} \right) \\ & + \sum_{k,I,\pm} \left(\frac{1}{2} \dot A^{kI\pm} \dot A^{kI\pm} - \frac{(k+1)^2}{2R^2} \mathrm{Tr}\, A^{kI\pm} A^{kI\pm} \right) \\ & - i \sum_{k,I,\pm} \left(\mathrm{Tr}\, \lambda_A^{kI\pm\dagger} \dot\lambda^{A,kI\pm} + \frac{k+\frac{3}{2}}{R} \mathrm{Tr}\, \lambda_A^{kI\pm\dagger} \lambda^{A,kI\pm} \right) \Bigg], \end{aligned}$$

where both properties of the harmonics and integration by parts were used. The transversality of the vector spherical harmonics, $\nabla^a Y_{(1)a}^{kJ\pm} = 0$, and the identity

$$\nabla_a \nabla_b Y_{(1)}^{iJ\pm a} = [\nabla_a, \nabla_b] Y_{(1)}^{iJ\pm a} = R_{cab}^a Y_{(1)}^{iJ\pm c} = \mathcal{R}_{ab} Y_{(1)}^{iJ\pm a} = \frac{2}{R^2} Y_{(1)b}^{iJ\pm}$$

is required in the computation of the masses for the vector modes.

A way of escalating up the different states of the Kaluza-Klein mass tower is by operating with the two supercharges $Q_L = (\mathbf{2}, \mathbf{1}, \bar{\mathbf{4}})$ and $Q_R = (\mathbf{1}, \mathbf{2}, \mathbf{4})$ which move you up the tower to the left and right respectively. These supercharges are representations that label states of $SU(2)_L \otimes SU(2)_R \otimes SU(4)$, i.e. the first number in the bracket of the supercharge representation labels a state in $SU(2)_L$, the second number labels a state in $SU(2)_R$, and the third number labels a state in $SU(4)$. Note that the superconformal transformations in the selected curved background geometry relates the whole tower of Kaluza-Klein modes. This means that the infinite tower of KK-modes is not decomposed into finite dimensional irreducible representations of the superconformal algebra. What we see instead is that the whole tower itself is a single superconformal irrep.

By only considering half of the supercharges, which we will denote by Q_L and with the addition of the bosonic symmetries we generate the subalgebra $SU(2|4)$. The KK modes are decomposed in terms of finite dimensional irreps of this group. Higher supermultiplets in the tower generally branch into 5 irreps under the bosonic subalgebra $SU(2) \otimes SU(4)$ which implies that all of them are short supermultiplets of $SU(2|4)$. An important result of the supersymmetry algebra is that the ground state energy, given as the sum of zero point energies, should vanish. An example of this is obtained by considering the lowest lying supermultiplet $(\mathbf{1},\mathbf{1},\mathbf{6}) + (\mathbf{2},\mathbf{1},\mathbf{4}) + (\mathbf{3},\mathbf{1},\mathbf{1})$, in which case we find

$$6 \cdot \frac{1}{R} - 8 \cdot \frac{3}{2R} + 3 \cdot \frac{2}{R} = \frac{6}{R} - \frac{12}{R} + \frac{6}{R} = 0.$$

The truncation to be performed is exactly the restriction to this lowest lying multiplet.

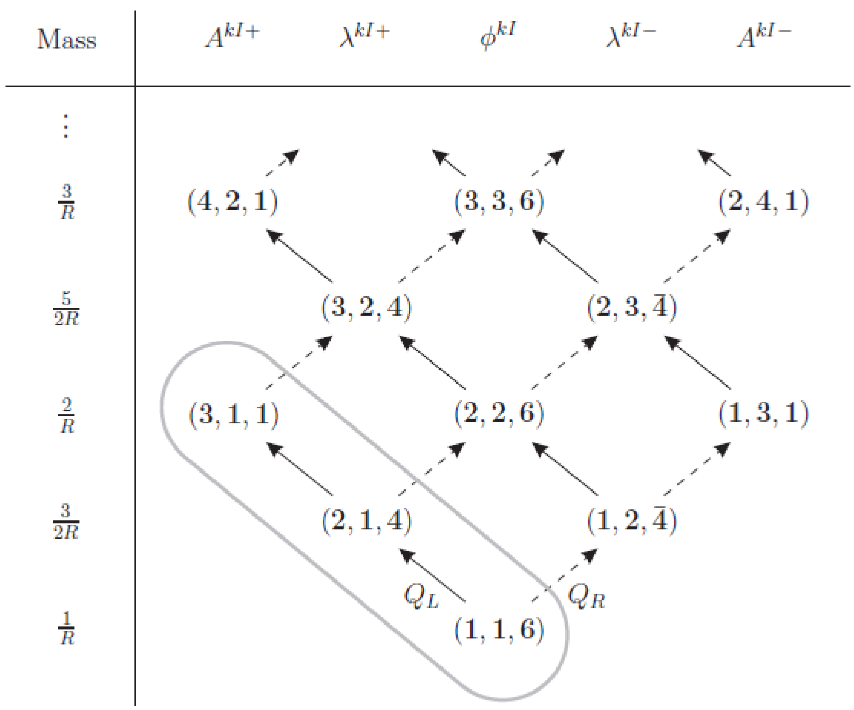


Figure 7: This Kaluza-Klein mass tower (which can be found in [13] at the end of section 2.1) demonstrates states that are labelled by representations of $SU(2)_L \otimes SU(2)_R \otimes SU(4)$. To climb upward to the left of the tower, one acts with the supercharge $Q_L = (\mathbf{2},\mathbf{1},\bar{\mathbf{4}})$, and to climb upward to the right, one acts with the supercharge $Q_R = (\mathbf{1},\mathbf{2},\mathbf{4})$. The states that are encircled are involved in the steady truncation to the plane-wave matrix theory.

3.8 Conclusion

This section started with motivation from the Fundamental Orthogonality relation, by performing Fourier expansions of finite groups to allow us to get the "position" space and "momentum" space functions of the group. Using the developed and necessary delta function for the finite group, the Fourier transform pairs constructed were found to be consistent. We next moved on to the description of a coset and argued as to why the coset $SO(3)/SO(2)$ is the 2-sphere, and we followed through with a similar argument as to why the coset $SO(4)/SO(3)$ is the 3-sphere. This lead us to determining the Fourier transformation on a coset which made use of a "projector" (which is actually an intertwining map) and the delta function for finite groups and then Lie groups. We find that the harmonics (scalar, vector, tensor, etc) that are used to perform the Fourier transformation are given by the matrix elements $D^n_{i\zeta,q}(g)$ - this leads to the construction of these spherical harmonics. We made use of techniques already known in enabling us to check first if we are able to obtain the spherical harmonics of the coset $SO(3)/SO(2)$, the 2-sphere, which we have succeeded in doing. From there we started on the construction of the spherical harmonics for the coset $SO(4)/SO(3)$, the 3-sphere. We constructed the scalar and vector harmonics for a few selected irreps ($(\frac{1}{2}, \frac{1}{2})$ and $(1, 1)$ to be specific) and we performed two convincing checks to see if what we constructed is valid; what we have found is that our results check out. Following after that, we reviewed the work of Kim, Klose, and Plefka [13] on the harmonic expansion of $\mathcal{N} = 4$ SYM on the 3-sphere to understand the spectrum of the theory. We think of the spectrum of the 3-sphere as the "fingerprint" of the theory as a way of identifying and knowing in great detail the nature of the giant graviton. In the chapters to follow, we will use the knowledge and results obtained in this section to see if the expected emergent Yang-Mills theory does indeed emerge.

4 Light-Front Partons and Dimensional Reduction in Relativistic Field Theory

4.1 Introduction

t'Hooft [14] has suggested that in order to harmonise gravity with quantum mechanics there is a necessary reduction needed of the degrees of freedom from $3+1$ to $2+1$ dimensions. The reduction in question, which operates as a holographic mapping, has to allow for an inverse process in which we can recover information of the $3D$ from the $2D$. A model was showcased by t'Hooft, based on homogeneous structures, as a realisation of this idea. Susskind has emphasised in a paper [15] the use of light-front quantisation and partons to realise the holographic projection. It is certain that a string theory can be developed in the light-cone frame without the need of a longitudinal dimension. Generally, one has the simple idea of light-front partons as the most crucial degrees of freedom.

The importance of light-front partons along with the affiliated dimensional reduction is already visible in the example of a relativistic scalar field. We will describe this at the classical level, with no interactions included. We pick this case since it illustrates the basic ideas in the simplest possible setting. From there, we are able to elaborate on a mechanism for gaining the complete theory from the lower dimensional picture (the parton picture).

The idea is to consider a relativistic field in the light-front frame. Once one compactifies the longitudinal direction, x^-, one truncates the fields by retaining only the oscillators that have the lowest value of p^+. These then are the essential partonic degrees of freedom. Is it possible to construct the complete theory from the partons? We will present the possibility of a $3+1$ Lorentz symmetry in the partonic $2+1$ dimensional theory. A set of generators are given that close a $4D$ Poincaré algebra. These have a connection to the generators of the $4D$ relativistic membrane; this connection will be explained in detail. Next, the discussion will move onto the main topics which focus on the N-body problem of free partons. It will be exposed that, by using well-defined fields, the complete $4D$ relativistic scalar field theory is reconstructed. The rebuilding is achieved by using a particular analysis of the N-parton configuration space. Instead of the usual treatment of identical particles, which forbids coincident particles, there is the inclusion of configurations of coincident partons. This is analogous to the analysis of light-cone strings [16],[17]. Since a relativistic field can be expressed in terms of non-relativistic partons of a lower dimensional theory, this construction is likely to give a fresh understanding into the properties of local field theory. This chapter is a review of [3].

4.2 Relativistic scalar field model

We consider a relativistic scalar field theory in $3+1$ dimensions whose Lagrangian density is [3]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\phi).$$

Considering the co-ordinate $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3)$ where on the light-front frame x^+ is a constant, the Dirac brackets we have are

$$\{\varphi(x^-, x), \varphi(y^-, y)\} = (\partial_-)^{-1}\delta(x^- - y^-)\cdot\delta(x-y),$$

where $x = (x^1, x^2)$ are the transverse directions (see appendix E for the in-depth computation of this bracket). The Hamiltonian, which generates translations in the x^+ direction, and the longitudinal momentum, which generates translations in the x^- direction, are [3]

$$\begin{aligned} H \equiv P^- &= \int dx^-\ d^2x \left[\frac{1}{2}\nabla\varphi\cdot\nabla\varphi + V(\varphi)\right] \\ P^+ &= \int dx^-\ d^2x(\partial_-\varphi)^2. \end{aligned}$$

By compactifying the x^- direction with a radius of $R = 2\pi$, the conjugate momenta is $p^+ = \frac{2\pi n}{R}(= n)$ and $n > 0$ in integer intervals. By expanding in the p^+ modes [3], we get

$$\varphi(x^-, x) = \sum_{n>0} \frac{1}{\sqrt{4\pi p_n^+}} \left(A_n(x)e^{-ip_n^+} + A_n^\dagger(x)e^{ip_n^+}\right) + \frac{1}{\sqrt{2\pi}}\tilde{\phi}(0, x).$$

To obtain the parton picture truncate the system to the $p^+ = 1$ mode. All other modes are dropped. After truncation, the mode expansion is

$$\varphi(x^-, x) \approx \frac{1}{\sqrt{4\pi}}\left(A_1(x)e^{-ix^-} + A_1^\dagger(x)e^{ix^-}\right),$$

and the Hamiltonian and the longitudinal momentum read

$$\begin{aligned} H &= \int d^2x \left(\frac{1}{2}\nabla A^\dagger(x)\nabla A(x) + V\right) \\ P^+ &= \int d^2x\ A^\dagger(x)A(x). \end{aligned}$$

In these expressions, the index $p^+ = 1$ has been dropped from the creation-annihilation operators $A^\dagger(x)$ and $A(x)$ respectively. Generally, if the reduction is to be interpreted as integrating out degrees of freedom, then there will be changes to the interaction term. $A^\dagger(x)$ and $A(x)$ obey the Poisson brackets of a non-relativistic second quantised Schrödinger field

$$\{A(x), A^\dagger(x')\} = \delta^{(2)}(x - x')$$

and the density function is given as $\rho(x) \equiv A^\dagger(x)A(x)$. From this, we see that by integrating the density function [3] we get the longitudinal momentum

$$P^+ = \int d^2x\ \rho(x) = \hat{N},$$

which is the number operator. After the change of variables

$$\begin{aligned} A(x) &= \sqrt{\rho}e^{i\pi} \\ A^\dagger(x) &= \sqrt{\rho}e^{-i\pi}, \end{aligned}$$

the Hamiltonian is [3]

$$H = \int d^2x \left(\frac{1}{2}\rho\nabla\pi\cdot\nabla\pi + V'(\rho)\right),$$

where the potential energy has gained an extra ρ-dependent term. In what is to follow, interactions and $\hbar$-quantum effects are ignored. In conclusion, when truncating the relativistic field to the point of only the $p^+ = 1$ mode, a non-relativistic type theory in $2+1$ dimensions with $\rho(x)$ and $\pi(x)$ as conjugate fields emerges.

A 4D Lorentz symmetry still operates in this reduced theory. This symmetry is given by the following set of generators [3]

$$\begin{aligned} J^{ab} &= \int d^2x\ \pi(x^a \partial_b - x^b \partial_a)\rho \\ J^{a-} &= \int d^2x\ \frac{1}{2}\Big(x^a \rho(\nabla\pi)^2 - \rho(\partial_a \pi^2)\Big) \\ J^{a+} &= \int d^2x\ x^a \rho \\ J^{+-} &= -\int d^2x\ \rho\pi, \end{aligned}$$

where $a, b = 1, 2$. Combining this set of generators together with [3]

$$\begin{aligned} P^- &= \int d^2x\ \frac{1}{2}\rho(\nabla\pi)^2 \\ P^+ &= \int d^2x\ \rho \\ P^a &= \int d^2x\ \pi\partial_a\rho \end{aligned}$$

one can verify that they close the $3+1$ dimensional Poincaré algebra.

4.3 N-body system

We will now demonstrate how to recover the complete relativistic field theory from the N-body system of partons. Consider the kinetic energy

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2,$$

with the coordinates being in $d = D - 2$ dimensions: $x_i = (x_i^1, x_i^2, \ldots, x_i^d)$. The null-front time $x^+ = x^0 + x^{d+1}$ is the time conjugate to the Hamiltonian, and so $H = P^-$. The total longitudinal momentum is

$$P^+ = \sum_{i=1}^{N} 1 = N.$$

An arrangement of cumulative fields $\rho_n(x)$ will now be defined on the coordinate space $X^d \times X^d \times \cdots \times X^d / S_N$ by interpreting this space as an orbifold.

The intention is to treat the configurations in a particular way when partons coincide. In the standard analysis of identical particles, one omits the points from the configuration space when particles are at the same point. These points will now be included and used to define new observables. Doing this introduces a chain of cumulative density fields as follows. Consider

$$\rho_1(x) = \sum_i{}' \delta(x - x_i)$$

where the prime signifies that we only have non-coincident coordinates in the sum. To achieve this, introduce an infinitesimal distance ϵ and a minimal separation by $|x_{i'} - x_{i''}| > \epsilon$. If two coordinates are within ϵ, it will be considered a composite object with $p^+ = 2$. All pairs of coordinates such that $|x_{i'} - x_{i''}| < \epsilon$ define a new field [3]

$$\rho_2(x) = \sum_{(i',i'')} \frac{1}{2} \left(\delta(x - x_i^{'}) + \delta(x - x_i^{''}) \right).$$

In general [3]

$$\rho_n(x) = \sum_{(i_1,i_2,\cdots,i_n)} \frac{1}{n} \left(\sum_{k=1}^{n} \delta(x - x_{i_k}) \right)$$

which receives a contribution only if n of the particles are at the same location. This defines the $p^+ = n$ density field. The new densities define the creation-annihilation operators in the expansion of the relativistic field $\varphi(x^-, x)$. To be more specific,

$$\rho_n(x) = A_n^\dagger(x) A_n(x).$$

Thus, the correct reconstruction of the total longitudinal momentum operator have been achieved:

$$P^+ = \sum_{n>0} n \int \rho_n(x) dx = \sum_{n>0} n \int A_n^\dagger A_n dx^d.$$

The Hamiltonian, however, it less simple. A method that will be used is

$$\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} = \sum_n \sum_i \int dxdy \frac{\partial \rho_n(x)}{\partial x_i} \frac{\partial \rho_n(y)}{\partial x_i} \frac{\partial}{\partial \rho_n(x)} \frac{\partial}{\partial \rho_n(y)} + \frac{\partial^2 \rho_n}{\partial x_i^2} \frac{\partial}{\partial \rho_n},$$

where the second term leads to a potential-type contributions so we will just consider the first kinetic term only. By performing a simple calculation, we find

$$\sum_i \rho_i^2 \to \int d^2x \sum_n \frac{1}{n} \rho_n(x) \nabla \pi_n(x) \nabla \pi_n(x) + \cdots$$

and combining it with

$$A_n = \sqrt{\rho_n} e^{i\pi_n}, \quad A_n^\dagger = \sqrt{\rho_n} e^{-i\pi_n}$$

we then have

$$P^- = \sum_i \frac{1}{2} p_i^2 \to \int d^dx \sum_n \frac{1}{2n} \nabla A_n^\dagger \nabla A_n = \int dx^- dx^d \nabla \varphi \nabla \varphi.$$

Thus, the relativistic field theory expression is recovered from the non-relativistic parton picture.

4.4 Conclusion

For a system of identical particles, we remove from the coordinate space configurations where any particles are coincident. This assumption is essential for anyon statistics in 2d to exist, for example. For the light front partons, configurations with coinciding particles have been retained and we have argued that they carry additional information. This information is the origin of density fields of increasing p^+, which reconstruct the single relativistic field. If partons obey usual common statistics, this construction would not be possible.

We will try to use the mechanism described in this chapter to explain how the emergent Yang-Mills theory is constructed. Basically, our partons will be the magnons of the spin chain description of the super Yang-Mills theory. The higher momentum modes will be obtained as bound states of the original magnons, providing a dynamical mechanism for the origin of coincident parton states. To develop this idea in detail, we need to review the magnon bound states of the super Yang-Mills theory. This is the topic of the next chapter.

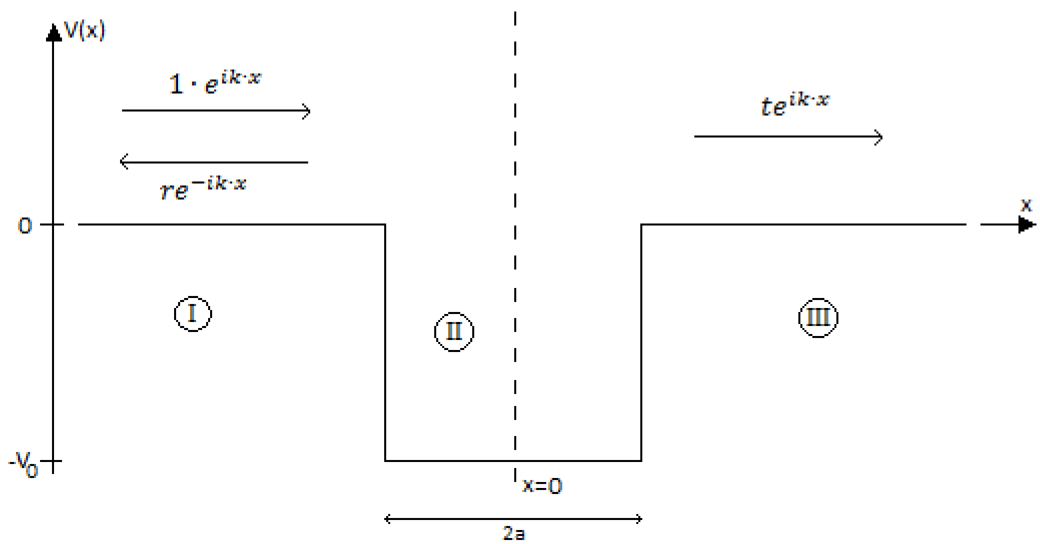
5 Magnon Bound States

5.1 Introduction

A very powerful approach to the planar limit of the super Yang-Mills theory is provided by the spin chain description of the CFT operators [18]. This description maps each CFT operator to a state of a spin chain and it maps the dilatation operator to the Hamiltonian of the spin chain. The dynamics of the spin chain is naturally described in terms of excitations known as magnons. The magnon scattering matrix is determined exactly by the symmetries of the problem. We will argue in the next section that poles in the scattering matrix are associated with bound states of the particles that are scattering. We will argue this in the setting of non-relativistic quantum mechanics where the discussion is simplest, but the conclusion is general. This connection will prove to be useful because, since we know the exact magnon S-matrix, it is possible to determine the exact magnon bound state spectrum. We will try to apply the mechanism of the previous chapter to the problem of understanding emergent Yang-Mills theory. We will identify magnons as the basic partons and magnon bound states with the coincident parton configurations. It is thus crucial to determine the spectrum of magnon bound states. This is one of the main accomplishments of this chapter.

5.2 Association between a pole and a bound state

The material in this section is standard non-relativistic quantum mechanics and appears, for example, in [19]. For propagating waves in the case of the finite square well, we have the following set up



where we have our incident wave of the form $1 \cdot e^{ik\cdot x}$ where the coefficient of the incident wave is chosen to be 1 so that conservation of probability implies the condition $1 = T + R$. This condition comes about since there are two outcomes for an incoming wave: it is either reflected back (with probability R) or it transmits through (with probability T). The sum of these two probabilities give you the probability of 1. The potential in region I and III is zero and in region II is equal to $-V_0$ where $V_0 > 0$. We are not too concerned about region II so we place our attention on regions I and III. In region III, we see that the wave has the same exponential form as the incident wave in region I, and the potential in region III obeys the condition $V(x)_{|x=\infty} = 0$.

For a bound state, we have the condition $E = \frac{k^2}{2m} < 0$. For the wave in region III ($te^{ik\cdot x}$) to die as $x \to \infty$, we must have $k = i\kappa$ where κ is positive and real. After this substitution, the wave ($te^{ik\cdot x} = te^{i(i\kappa)\cdot x} = te^{-\kappa\cdot x}$) certainly does die as $x \to \infty$. In region I, as $x \to -\infty$, the reflected wave $re^{-ik\cdot x} = re^{\kappa\cdot x}$ dies. However, the incident wave $1 \cdot e^{ik\cdot x} = 1 \cdot e^{-\kappa\cdot x}$ blows up as $x \to -\infty$. To fix this problem, one would intuitively set the coefficient of the wave to zero, yet in this case you cannot since the coefficient has already been set to 1. Thus, to make this term *seem* as it is going to zero is to set the reflective wave's coefficient r to ∞ since the full equation of region I is $1 \cdot e^{-\kappa\cdot x} + re^{\kappa\cdot x}$.

To quickly revise, for the bound state ($E < 0$), with the momentum being purely imaginary ($k = i\kappa$) and respecting the condition that waves die at large distances, we find a pole in the reflection coefficient r. Thus, a pole in r is associated to a bound state.

To illustrate this argument, we will first calculate the bound state energies. For the finite square well potential (see section 2.6 in [19]), the wave function has the following forms:

$$\psi(x) = \begin{cases} Ae^{ik_1 x} + Be^{-ik_1 x}, & x < -a \\ C\sin(k_2 x) + D\cos(k_2 x), & -a \leq x \leq a \\ Fe^{ik_1 x} & x > a \end{cases}$$

where $k_1 = \frac{\sqrt{2mE}}{\hbar}$ and $k_2 = \frac{\sqrt{2m(E+V_0)}}{\hbar}$. At $x = -a$, the continuity conditions imply

- $Ae^{-ik_1 a} + Be^{ik_1 a} = C\cos(k_2 a) - D\sin(k_2 a)$,
- $ik_1(Ae^{-ik_1 a} - Be^{ik_1 a}) = k_2\,(C\sin(k_2 a) + D\cos(k_2 a))$.

At $x = a$, the continuity conditions imply [19]

- $C\cos(k_2 a) + D\sin(k_2 a) = Fe^{ik_1 a}$,
- $k_2\,(-C\sin(k_2 a) + D\cos(k_2 a)) = ik_1 Fe^{ik_1 a}$.

Through further manipulations of these equations we find [19]

$$F = \frac{2k_1 k_2 e^{-2ik_1 a}}{2k_2 k_1 \cos(2k_2 a) - i\sin(2k_2 a)(k_1^2 + k_2^2)} A,$$

and thus the transmission coefficient is compute to be [19]

$$T = \left|\frac{F}{A}\right|^2 = \frac{4k_1^2 k_2^2}{\sin^2(2k_2 a)(k_1^2 - k_2^2)^2 + 4k_1^2 k_2^2} = \frac{1}{1 + (k_1^2 - k_2^2)^2 \left[\frac{\sin^2(2k_2 a)}{4k_1^2 k_2^2}\right]}.$$

Then, the reflection coefficient is calculated to be

$$\begin{aligned} R = 1 - T = 1 - \frac{1}{1 + (k_1^2 - k_2^2)^2 \left[\frac{\sin^2(2k_2 a)}{4k_1^2 k_2^2}\right]} &= -\frac{(k_1^2 - k_2^2)^2 \left[\frac{\sin^2(2k_2 a)}{4k_1^2 k_2^2}\right]}{1 + (k_1^2 - k_2^2)^2 \left[\frac{\sin^2(2k_2 a)}{4k_1^2 k_2^2}\right]} \\ &= -\frac{(k_1^2 - k_2^2)^2 \sin^2(2k_2 a)}{4k_1^2 k_2^2 + (k_1^2 - k_2^2)^2 \sin^2(2k_2 a)}. \end{aligned}$$

Poles in this coefficient appear when

$$4k_1^2k_2^2 + (k_1^2 - k_2^2)^2 \sin^2(2k_2a) = 0,$$

and solving for k_1 we get

$$\begin{aligned}
k_1 &= \pm i k_2 \cot(k_2a) \quad , \quad k_1 = \pm i k_2 \tan(k_2a) \\
\Rightarrow k_1^2 &= -k_2^2 \cot^2(k_2a) \quad , \quad k_1^2 = -k_2^2 \tan^2(k_2a) \\
\Rightarrow \frac{2mE}{\hbar^2} = -\frac{2m(E+V_0)}{\hbar^2} \cot^2\left(\frac{\sqrt{2m(E+V_0)}}{\hbar}a\right) &\quad , \quad \frac{2mE}{\hbar^2} = -\frac{2m(E+V_0)}{\hbar^2} \tan^2\left(\frac{\sqrt{2m(E+V_0)}}{\hbar}a\right) \\
\Rightarrow \frac{-E}{E+V_0} = \cot^2\left(\frac{\sqrt{2m(E+V_0)}}{\hbar}a\right) &\quad , \quad \frac{-E}{E+V_0} = \tan^2\left(\frac{\sqrt{2m(E+V_0)}}{\hbar}a\right).
\end{aligned}$$

We stop here and now proceed to solve for the bound state energies. Considering the 4 equations from the continuity condition for $x = -a$ and $x = a$, by dividing the second equation by the first (B is set to zero) we find

$$ik_1 = k_2 \left[\frac{C\sin(k_2a) + D\cos(k_2a)}{-C\cos(k_2a) + D\sin(k_2a)}\right]$$

and by dividing the fourth equation by the third implies

$$ik_1 = k_2 \left[\frac{-C\sin(k_2a) + D\cos(k_2a)}{C\cos(k_2a) + D\sin(k_2a)}\right].$$

The only way that these two equations for ik_1 can be consistent is if one of the constants C or D vanish:

- $C \neq 0$: $ik_1 = -k_2 \tan(k_2a)$, even parity.
- $D \neq 0$: $ik_1 = k_2 \cot(k_2a)$, odd parity.

These equations can be rewritten as

$$\begin{aligned}
(ik_1)^2 &= -k_1^2 = (-k_2\tan(k_2a))^2 = k_2^2\tan^2(k_2a) \qquad \textbf{even parity} \\
(ik_1)^2 &= -k_1^2 = (k_2\cot(k_2a))^2 = k_2^2\cot^2(k_2a) \qquad \textbf{odd parity} \\
\Rightarrow -\frac{2mE}{\hbar^2} &= \frac{2m(E+V_0)}{\hbar^2}\tan^2\left(\frac{\sqrt{2m(E+V_0)}}{\hbar}a\right) \\
-\frac{2mE}{\hbar^2} &= \frac{2m(E+V_0)}{\hbar^2}\cot^2\left(\frac{\sqrt{2m(E+V_0)}}{\hbar}a\right) \\
\Rightarrow -\frac{E}{E+V_0} &= \tan^2\left(\frac{\sqrt{2m(E+V_0)}}{\hbar}a\right) \\
-\frac{E}{E+V_0} &= \cot^2\left(\frac{\sqrt{2m(E+V_0)}}{\hbar}a\right),
\end{aligned}$$

which reproduces the results we obtained for the location of poles in the reflection coefficient R. This nicely illustrates the fact that poles in the reflection coefficient are associated to bound states.

5.3 Review of Magnons

In this section we will review how the magnon scattering matrix is determined completely by the symmetries of the problem. This scattering matrix will then allow us to determine the exact bound state spectrum. We start by describing, in detail, the representation relevant for a single magnon. We then use this representation to show that the scattering matrix S is completely determined, up to an arbitrary overall phase.

The symmetry of the sector of the gauge theory considered is $SU(2|2) \times SU(2|2)$ (which can also be denoted as $SU(2|2)^2$). The $SU(2|2)$ algebra we are interested in is

$$\{Q_a^\alpha, Q_b^\beta\} = \epsilon^{\alpha\beta}\epsilon_{ab}\frac{k}{2}$$

$$\{S_\alpha^a, S_\beta^b\} = \epsilon^{ab}\epsilon_{\alpha\beta}\frac{k^*}{2}$$

$$\{S_\alpha^a, Q_b^\beta\} = \delta_b^a L_\alpha^\beta + \delta_\alpha^\beta R_b^a + \delta_b^a \delta_\alpha^\beta C.$$

There are commutators for the bosonic generators as well. We note that there are three central charges for each $SU(2|2)$ factor. However, following [20] (see discussion around (3.13)) we will set the central charges of the two copies to be equal. Physical states of the super Yang-Mills theory have vanishing central charges. A question that comes to mind is what are the two $SU(2|2)$ factors? To answer this let us consider the bosonic fields. There are 6 hermitian adjoint scalars ϕ^i that transform as a vector of $SO(6)$. They can be combined to form the complex fields as follows

$$\begin{aligned} X &= \phi^1 + i\phi^2 \qquad & \bar{X} &= \phi^1 - i\phi^2 \\ Y &= \phi^3 + i\phi^4 \qquad & \bar{Y} &= \phi^3 - i\phi^4 \\ Z &= \phi^5 + i\phi^6 \qquad & \bar{Z} &= \phi^5 - i\phi^6. \end{aligned}$$

The symmetry group that we study transforms Y,X,$\bar{Y}$ and $\bar{X}$. However, it does not act on Z and $\bar{Z}$. Thus, the bosonic piece of the symmetry group is the $SO(4)$ that rotates ϕ^1,ϕ^2,ϕ^3 and ϕ^4 as a vector. Recall that $SO(4)$ is equivalent to $SU(2)_L \times SU(2)_R$. We will explicitly describe a useful relation between the $SO(4)$ generators (M_{pq}) and the $SU(2)$ generators (J_a^L, J_a^R). We will use the normalisations of the $SU(2)$ generators as

$$[J_3, J_\pm] = \pm 2J_\pm \qquad [J_+, J_-] = J_3.$$

In this normalisation, J_3 is always an integer for finite dimensional irreps. The generators of the left and right $SU(2)$ are

$$\begin{aligned} J_3^L &= -i(M_{12} + M_{34}) \\ J_+^L &= -\frac{1}{2}((M_{13} - M_{24}) + i(M_{14} + M_{23})) \\ J_-^L &= \frac{1}{2}((M_{13} - M_{24}) - i(M_{14} + M_{23})) \\ J_3^R &= -i(M_{12} - M_{34}) \\ J_+^R &= \frac{1}{2}(-(M_{13} + M_{24}) + i(M_{14} - M_{23})) \\ J_-^R &= \frac{1}{2}((M_{13} + M_{24}) + i(M_{14} - M_{23})). \end{aligned}$$

One way to understand these formulae is to use $M_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$ and then change to complex variables:

$$
\begin{aligned}
z_1 &= x_1 + i x_2 \\
z_2 &= x_3 + i x_4 \\
\bar{z}_1 &= x_1 - i x_2 \\
\bar{z}_2 &= x_3 - i x_4 .
\end{aligned}
$$

In terms of the complex variables, the $SU(2)_L \times SU(2)_R$ generators read

$$
\begin{aligned}
J_3^L &= z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \\
J_3^R &= z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \\
J_+^L &= z_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial \bar{z}_2} \\
J_-^L &= \bar{z}_1 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial z_1} \\
J_+^R &= \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial z_2} \\
J_-^R &= \bar{z}_1 \frac{\partial}{\partial \bar{z}_2} - z_2 \frac{\partial}{\partial z_1} .
\end{aligned}
$$

The (J_3^L, J_3^R) charges are

$$
z_1 \to (1,1), \qquad z_2 \to (1,-1), \qquad \bar{z}_2 \to (-1,1), \qquad \bar{z}_1 \to (-1,-1).
$$

These charges show that the vector of $SO(4)$ transforms as $\left(\frac{1}{2}, \frac{1}{2}\right)$ of $SU(2)_L \times SU(2)_R$. In our normalisation, the usual $\frac{1}{2}$ has become 1. Given these charges, the individual terms in $J_\pm^{L,R}$ are clear. Since the transformation of z_1 is the same as that of X and the transformation of z_2 is the same as that of Y, we have understood (3.13) of [20]. So the states $|\phi^a\rangle$ and $|\psi^a\rangle$ that we use to discuss the $SU(2|2)$ symmetry are not easily related to fields of the super Yang-Mills theory - it is products of pairs of these states that are related.

According to our above discussion, each magnon will transform as some $SU(2|2)^2$ representation, labelled by some central charges, which we need to determine. Note that we also need to give the $SU(2)$ representation of each of the $SU(2)$ factors in $SU(2|2)$ - these are both spin $\frac{1}{2}$. To specify the representation that each magnon transforms in, we need to specify an a,b,c, and d for each, where

$$
\begin{aligned}
Q_a^\alpha |\phi^b\rangle &= a \delta_a^b |\psi^\alpha\rangle \qquad & Q_a^\alpha |\psi^\beta\rangle &= b \epsilon^{\alpha\beta} \epsilon_{ab} |\phi^b Z^+\rangle \\
S_\alpha^a |\phi^b\rangle &= c \epsilon_{\alpha\beta} \epsilon^{ab} |\psi^\beta Z^-\rangle \qquad & S_\alpha^a |\psi^\beta\rangle &= d \delta_\alpha^\beta |\phi^a\rangle .
\end{aligned}
$$

Here is an explanation of the notation: when we say we have given a magnon a momentum, we mean we have performed the sum

$$
\sum_n e^{ipn} \cdots ZZZYZZZ \cdots
$$

where the Y appears in the n^{th} gap between the Z's. We introduce the notation

$$|Z^{\pm}Y\rangle = \sum_n e^{ipn} \cdots ZZZYZZZ \cdots$$

where now the Y on the right hand side sits in the $(n \pm 1)^{\text{th}}$ gap between the Z's. Thus, we can relabel n such that

$$|Z^{\pm}Y\rangle = \sum_n e^{ip(n\pm 1)} \cdots ZZZYZZZ \cdots$$

where now Y again sits in the n^{th} position. We can write this as

$$|Z^{\pm}Y\rangle = e^{\mp ip}|YZ^{\pm}\rangle.$$

Using this notation, we have

$$Q^1_1 Q^2_2|\phi^2\rangle = aQ^1_1|\psi^2\rangle = ba\epsilon^{12}\epsilon_{12}|\phi^2 Z^+\rangle$$

and

$$Q^2_2 Q^1_1|\phi^2\rangle = 0.$$

Using these two results and recalling that

$$Q^{\alpha}_a Q^{\beta}_b + Q^{\beta}_b Q^{\alpha}_a = \epsilon^{\alpha\beta}\epsilon_{ab}\frac{k}{2},$$

we learn that

$$k = 2ab.$$

By a very similar argument using the S^a_α supercharges, we find

$$k^* = 2cd.$$

This can now be extended to the case where our state has k magnons

$$\{Q^{\alpha}_a, Q^{\beta}_b\}|M_1 M_2 \cdots M_K\rangle = \sum_{k=1}^{K} a_k b_k \prod_{l=k+1}^{K} e^{-ip_l}|M_1 M_2 \cdots M_K\rangle.$$

When the supercharges act on the k^{th} magnon, they insert a Z into the chain. Thus, all the magnons M_i with $i \geq k$ must be shifted by one position and this leads to the phase $\prod_{l=k+1}^{K} e^{-ip_l}$ on the right hand side of the expression above. The physical K magnon state must have vanishing central charges. If we are to obtain a representation without central extension, we must require that the central charge vanishes

$$\frac{k}{2} = \sum_{k=1}^{K} a_k b_k \prod_{l=k+1}^{K} e^{-ip_l} = 0. \tag{12}$$

Recall that our operators vanish unless the phases $q_k = e^{ip_k}$ obey

$$\prod_k q_k = 1 \Rightarrow \sum_k p_k = 0.$$

In this case (12) is solved by

$$a_k b_k = \alpha(e^{-ip_k} - 1).$$

The proof follows as (this proof is given in [20])

$$\begin{aligned}\frac{k}{2} &= \sum_{k=1}^{K} a_k b_k \prod_{l=k+1}^{K} e^{-ip_l} \\ &= \alpha \sum_{k=1}^{K} (e^{ip_k} - 1) \prod_{l=k+1}^{K} e^{-ip_l}.\end{aligned}$$

To motivate what comes next, consider a specific example. For $K = 4$, we have

$$\begin{aligned}\alpha \sum_{k=1}^{4} (e^{ip_k} - 1) \prod_{l=k+1}^{4} e^{-ip_l} &= \alpha(e^{-ip_1} - 1)e^{-i(p_2+p_3+p_4)} + \alpha(e^{-ip_2} - 1)e^{-i(p_3+p_4)} + \\ &+ \alpha(e^{-ip_3} - 1)e^{-ip_4} + \alpha(e^{-ip_4} - 1) \\ &= e^{-i(p_1+p_2+p_3+p_4)} - 1.\end{aligned}$$

To get the final answer, notice that the second and third, third and fourth, and the fifth and sixth terms cancel in the second last expression. Thus

$$\frac{k}{2} = \alpha(e^{-i\sum_k p_k} - 1) = 0$$

completing the proof. A very similar argument using S^a_α supercharges shows that

$$c_k d_k = \beta(e^{ip_k} - 1).$$

Returning to the one magnon case, we easily find that

$$\begin{aligned}Q^\alpha_a S^b_\beta |\phi^c\rangle &= c Q^\alpha_a \epsilon^{bc} \epsilon_{\beta\gamma} |\psi^\gamma Z^-\rangle \\ &= cb \epsilon^{bc} \epsilon_{\beta\gamma} \epsilon^{\alpha\gamma} \epsilon_{ad} |\phi^d\rangle.\end{aligned}$$

Setting $a = b$ and $\alpha = \beta$, and summing over both indices, we find

$$Q^\alpha_a S^a_\alpha |\phi^c\rangle = 2bc|\phi^c\rangle.$$

Through very similar manipulations, it can be shown that

$$S^a_\alpha Q^\alpha_a |\phi^c\rangle = 2ad|\phi^c\rangle,$$

so that we learn that

$$\{Q^\alpha_a, S^a_\alpha\}|\phi^c\rangle = 2(ad + bc)|\phi^c\rangle.$$

Thus, we find

$$C = \frac{1}{2}(ad + bc).$$

From

$$\{S^a_\alpha, Q^\beta_b\} = \delta^a_b L^\beta_\alpha + \delta^\beta_\alpha R^a_b + \delta^a_b \delta^\beta_\alpha C,$$

we know that

$$\{S^1_2, Q^1_1\} = L^1_2.$$

Also, from

$$L^\alpha_\beta |\psi^\gamma\rangle = \delta^\gamma_\beta |\psi^\alpha\rangle - \frac{1}{2}\delta^\alpha_\beta |\psi^\gamma\rangle,$$

we know that

$$L^1_2 |\psi^2\rangle = |\psi^1\rangle.$$

We now easily find

$$\begin{aligned} \{S^1_2, Q^1_1\}|\psi^2\rangle &= S^1_2 Q^1_1 |\psi^2\rangle + Q^1_1 S^1_2 |\psi^2\rangle \\ &= dQ^1_1 |\phi^1\rangle + b\epsilon^{12}\epsilon_{12} S^1_2 |\phi^2 Z^+\rangle \\ &= ad|\psi^1\rangle + bc\epsilon^{12}\epsilon_{12}\epsilon^{12}\epsilon_{21}|\psi^1\rangle \\ &= (ad - bc)|\psi^1\rangle, \end{aligned}$$

so that we learn $ad - bc = 1$. This is also the condition to get an atypical representation of $su(2|2)$ - see comment in (2.25) of [20]. Now,

$$\begin{aligned} a_k b_k c_k d_k &= \alpha\beta(e^{-ip_k} - 1)(e^{ip_k} - 1) \\ &= 4\alpha\beta \sin^2 \frac{p_k}{2} \\ &= \frac{1}{4}\Big[(a_k d_k + b_k c_k)^2 - (a_k d_k - b_k c_k)^2\Big] \\ &= \frac{1}{4}\Big[(2C_k)^2 - 1\Big], \end{aligned}$$

where

$$C_k = \pm\sqrt{\frac{1}{4} + 4\alpha\beta \sin^2 \frac{p_k}{2}}.$$

The components of an energy eigenstate in different asymptotic regions are related by the bulk scattering matrix S. S must commute with the $su(2|2)$ group. The labels of the representations of individual magnons can change under the scattering, but they must do so in a way that preserves the central charges of the total state.

A useful parametrisation for the parameters of the representation is given [20]

$$\begin{aligned} a &= \sqrt{g}\eta & b &= \frac{\sqrt{g}}{\eta} f\left(1 - \frac{x^+}{x^-}\right) \\ c &= \frac{\sqrt{g} i\eta}{f x^+} & d &= \frac{\sqrt{g} x^+}{i\eta} f\left(1 - \frac{x^-}{x^+}\right). \end{aligned}$$

The parameters $x^{\pm}$ are set by the momentum of the magnon

$$e^{i\frac{p}{2\pi J}} = \frac{x^+}{x^-}. \tag{13}$$

The condition $ad - bc = 1$ to get an atypical representation implies that

$$x^+ + \frac{g^2}{2x^+} - x^- - \frac{g^2}{2x^-} = i,$$

which will be useful for what is to come below.

Consider the scattering matrix S_{12} for two bosonic magnon states $|\phi_1^a \phi_2^b\rangle$. The indices a and b are $su(2)$ indices. Since the S-matrix has to commute with the bosonic $su(2)$ generators, we know that it must be proportional to the identity in each given irrep of $su(2)$. Thus, we can break this bosonic state up into $su(2)$ irreps and the S-matrix must be proportional to the identity within each irrep. These are three possible terms - we can symmetrise the indices a and b (spin 1 piece), we can anti-symmetrise the indices a and b (spin 0 piece), or we can have something proportional to ϵ^{ab} (another spin 0 piece). Thus, we can write

$$S_{12}|\phi_1^a \phi_2^b\rangle = A_{12}|\phi_2^{\{a}\phi_1^{b\}}\rangle + B_{12}|\phi_2^{[a}\phi_1^{b]}\rangle + \frac{1}{2}C_{12}\epsilon^{ab}\epsilon_{\alpha\beta}|\psi_2^\alpha \psi_1^\beta Z^-\rangle. \tag{14}$$

The Z^{-1} above can be understood using dimensional analysis. A similar argument for the other $su(2)$ group gives

$$S_{12}|\psi_1^\alpha \psi_2^\beta\rangle = D_{12}|\psi_2^{\{\alpha}\psi_1^{\beta\}}\rangle + E_{12}|\psi_2^{[\alpha}\psi_1^{\beta]}\rangle + \frac{1}{2}F_{12}\epsilon_{ab}\epsilon^{\alpha\beta}|\phi_2^a \phi_1^b Z^+\rangle.$$

Finally, we also have

$$\begin{aligned} S_{12}|\phi_1^a \psi_2^\beta\rangle &= G_{12}|\psi_2^\beta \phi_1^a\rangle + H_{12}|\phi_2^a \psi_1^\beta\rangle \\ S_{12}|\psi_1^\alpha \phi_2^b\rangle &= K_{12}|\psi_2^\alpha \phi_1^b\rangle + L_{12}|\phi_2^b \psi_1^\alpha\rangle. \end{aligned}$$

As we work with these formulas, we will make them more and more explicit. To start, put $a = b = 1$ in equation (14). The above equation becomes

$$S_{12}|\phi_1^1 \phi_2^1\rangle = A_{12}|\phi_1^1 \phi_2^1\rangle.$$

Acting on this with Q_1^α, one finds

$$\begin{aligned} Q_1^\alpha S_{12}|\phi_1^1 \phi_2^1\rangle &= A_{12} Q_1^\alpha |\phi_1^1 \phi_2^1\rangle \\ &= A_{12}\big(a_2|\psi_2^\alpha \phi_1^1\rangle + a_1|\psi_2^1 \psi_1^\alpha\rangle\big). \end{aligned}$$

Now, reverse the order of these two actions

$$\begin{aligned} S_{12} Q_1^\alpha |\phi_1^1 \phi_2^1\rangle &= S_{12}(a_1|\psi_1^\alpha \phi_2^1\rangle + |\phi_1^1 \psi_2^\alpha\rangle) \\ &= a_1(K_{12}|\psi_2^\alpha \phi_1^1\rangle + L_{12}|\phi_2^1 \psi_1^\alpha\rangle) + a_2(G_{12}|\psi_2^\alpha \phi_1^1\rangle + H_{12}|\phi_2^1 \psi_1^\alpha\rangle). \end{aligned}$$

The requirement that Q_1^α commutes with S_{12} thus implies that

$$\begin{aligned} A_{12} a_2 &= a_1 K_{12} + a_2 G_{12} \\ A_{12} a_1 &= a_1 L_{12} + a_2 H_{12}. \end{aligned}$$

Next consider

$$S_{12}|\phi_1^1\phi_2^2\rangle = \frac{A_{12}}{2}\left(|\phi_2^1\phi_1^2\rangle + |\phi_2^2\phi_1^1\rangle\right) - \frac{B_{12}}{2}\left(|\phi_2^1\phi_1^2\rangle - |\phi_2^2\phi_1^1\rangle\right) + \frac{C_{12}}{2}\epsilon^{12}\epsilon_{\alpha\beta}|\psi_2^\alpha\psi_1^\beta Z^-\rangle.$$

Thus

$$\begin{aligned}
Q_1^1 S_{12}|\phi_1^1\phi_2^2\rangle &= \frac{A_{12}}{2}\left(a_2|\psi_2^1\phi_1^2\rangle + a_1|\phi_2^2\psi_1^1\rangle\right) + \frac{B_{12}}{2}\left(a_2|\psi_2^1\phi_1^2\rangle - a_1|\phi_2^2\psi_1^1\rangle\right) + \\
&+ \frac{C_{12}}{2}\left(b_2\epsilon^{12}\epsilon_{21}\epsilon_{12}\epsilon_{12}|\phi_2^2 Z^+\psi_1^1 Z^-\rangle - b_1\epsilon^{12}\epsilon_{12}\epsilon^{12}\epsilon_{12}|\psi_2^1\phi_1^2\rangle\right) \\
&= \frac{A_{12}}{2}\left(a_2|\psi_2^1\phi_1^2\rangle + a_1|\phi_2^2\psi_1^1\rangle\right) + \frac{B_{12}}{2}\left(a_2|\psi_2^1\phi_1^2\rangle - a_1|\phi_2^2\psi_1^1\rangle\right) - \\
&- \frac{C_{12}}{2}\left(b_2|\phi_2^2 Z^+\psi_1^1 Z^-\rangle + b_1|\psi_2^1\phi_1^2\rangle\right).
\end{aligned}$$

The sign in the very last term above is because we had to pull the supercharge past a fermion and that costs a minus sign. Now act in the opposite order

$$\begin{aligned}
S_{12}Q_1^1|\phi_1^1\phi_2^2\rangle &= S_{12}a_1|\psi_1^1\phi_2^2\rangle \\
&= a_1\left[K_{12}|\psi_2^1\phi_1^2\rangle + L_{12}|\phi_2^2\psi_1^1\rangle\right].
\end{aligned}$$

We thus find

$$\begin{aligned}
\frac{a_2A_{12}}{2} + \frac{a_2B_{12}}{2} - \frac{b_1C_{12}}{2} &= a_1K_{12} \\
\frac{a_2A_{12}}{2} - \frac{a_2B_{12}}{2} - \frac{b_1C_{12}}{2}\frac{x_1^-}{x_1^+} &= a_1L_{12}.
\end{aligned}$$

Next, consider

$$\begin{aligned}
Q_1^2 S_{12}|\psi_1^1\psi_2^1\rangle &= D_{12}\epsilon^{21}\epsilon_{12}(b_2|\phi_2^2Z^+\psi_1^1\rangle - b_1|\psi_2^1\phi_1^2Z^+) \\
&= D_{12}\epsilon^{21}\epsilon_{12}(b_2\frac{x_1^-}{x_1^+}|\phi_2^2\psi_1^1Z^+\rangle - b_1|\psi_2^1\phi_1^2Z^+).
\end{aligned}$$

By acting in the opposite order, we find

$$\begin{aligned}
S_{12}Q_1^2|\psi_1^1\psi_2^1\rangle = \epsilon^{21}\epsilon_{12}\Big(b_1\frac{x_2^-}{x_2^+}&\big(G_{12}|\psi_2^1\phi_1^2Z^+\rangle + H_{12}|\phi_2^2\psi_1^1\rangle\big) - \\
&- b_2\big(K_{12}|\psi_2^1\phi_1^2Z^+\rangle + L_{12}|\phi_2^2\psi_1^1Z^+\rangle\big)\Big).
\end{aligned}$$

We thus find

$$\begin{aligned}
b_2D_{12}\frac{x_1^-}{x_1^+} &= b_1\frac{x_2^-}{x_2^+}H_{12} - b_2L_{12} \\
-b_1D_{12} &= b_1\frac{x_2^-}{x_2^+}G_{12} - b_2K_{12}.
\end{aligned}$$

Comparing

$$\begin{aligned}
Q_1^2S_{12}|\psi_1^1\psi_2^2\rangle = \frac{D_{12}}{2}&\big(b_2\epsilon^{21}\epsilon_{12}|\phi_2^2Z^+\psi_1^2\rangle - b_1\epsilon^{21}\epsilon_{12}|\psi_2^2\phi_1^2Z^+\rangle\big) + \\
&+ \frac{E_{12}}{2}\big(b_2\epsilon^{21}\epsilon_{12}|\phi_2^2Z^+\psi_1^2\rangle + b_1\epsilon^{21}\epsilon_{12}|\psi_2^2\phi_1^2Z^+\rangle\big) \\
&+ \frac{F_{12}}{2}\epsilon^{12}\big(a_2\epsilon_{12}|\psi_2^2\phi_1^2Z^+\rangle + a_1\epsilon_{21}|\phi_2^2\psi_1^2Z^+\rangle\big)
\end{aligned}$$

and

$$S_{12}Q_1^2|\psi_1^1\psi_2^2\rangle = b_1\epsilon^{21}\epsilon_{12}\Big(G_{12}|\psi_2^2\phi_1^2 Z^+\rangle + H_{12}|\phi_2^2\psi_1^2 Z^+\rangle\Big)\frac{x_2^-}{x_2^+},$$

we learn that

$$\begin{aligned}
-b_1\frac{D_{12}}{2} + b_1\frac{E_{12}}{2} - \frac{F_{12}}{2}a_2 &= b_1 G_{12}\frac{x_2^-}{x_2^+} \\
b_2\frac{D_{12}}{2}\frac{x_1^-}{x_1^+} + b_2\frac{E_{12}}{2}\frac{x_1^-}{x_1^+} + a_1\frac{F_{12}}{2} &= b_1 H_{12}\frac{x_2^-}{x_2^+}.
\end{aligned}$$

Comparing

$$S_1^2 S_{12}|\phi_1^1\phi_2^1\rangle = A_{12}\Big(c_2\epsilon_{12}\epsilon^{21}|\psi_2^2 Z^-\phi_1^1\rangle + c_1\epsilon_{12}\epsilon^{21}|\phi_2^1\psi_1^2 Z^-\rangle\Big)$$

and

$$\begin{aligned}
S_{12}S_1^2|\phi_1^1\phi_2^1\rangle = c_1\epsilon_{12}\epsilon^{21}&\Big(K_{12}|\psi_2^2\phi_1^1 Z^-\rangle + L_{12}|\phi_2^1\psi_1^2 Z^-\rangle\Big)\frac{x_2^+}{x_2^-} + \\
&+ c_2\epsilon_{12}\epsilon^{21}\Big(G_{12}|\psi_2^2\phi_1^1 Z^-\rangle + H_{12}|\phi_2^1\psi_1^2 Z^-\rangle\Big),
\end{aligned}$$

we learn that

$$\begin{aligned}
c_1 A_{12} &= c_1 L_{12}\frac{x_2^+}{x_2^-} + c_2 H_{12} \\
c_1 A_{12}\frac{x_1^+}{x_1^-} &= c_1 K_{12}\frac{x_2^+}{x_2^-} + c_2 G_{12}.
\end{aligned}$$

The equations above can now be solved to determine the elements of the scattering matrix. The result is

$$\begin{aligned}
A_{12} &= S_{12}^0\frac{x_2^+ - x_1^-}{x_2^- - x_1^+} \\
B_{12} &= S_{12}^0\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\left(1 - 2\frac{1 - \frac{1}{x_2^- x_1^+}}{1 - \frac{1}{x_2^- x_1^-}}\frac{x_2^+ - x_1^+}{x_2^+ - x_1^-}\right) \\
C_{12} &= S_{12}^0\frac{2g^2\eta_1\eta_2}{f x_1^+ x_2^+}\frac{1}{1 - \frac{1}{x_1^+ x_2^+}}\frac{x_2^- - x_1^-}{x_2^- - x_1^+} \\
D_{12} &= -S_{12}^0 \\
E_{12} &= -S_{12}^0\left(1 - 2\frac{1 - \frac{1}{x_2^+ x_1^-}}{1 - \frac{1}{x_2^- x_1^-}}\frac{x_2^- - x_1^+}{x_2^- - x_1^+}\right) \\
F_{12} &= -S_{12}^0\frac{2f(x_1^+ - x_1^-)(x_2^+ - x_2^-)}{\eta_1\eta_2 x_1^- x_2^-}\frac{1}{1 - \frac{1}{x_1^- x_2^-}}\frac{x_2^+ - x_1^+}{x_2^- - x_1^+} \\
G_{12} &= S_{12}^0\frac{x_2^+ - x_1^+}{x_2^- - x_1^+} \qquad H_{12} = S_{12}^0\frac{\eta_1}{\eta_2}\frac{x_2^+ - x_2^-}{x_2^- - x_1^+} \\
K_{12} &= S_{12}^0\frac{\eta_2}{\eta_1}\frac{x_1^+ - x_1^-}{x_2^- - x_1^+} \qquad L_{12} = S_{12}^0\frac{x_2^- - x_1^-}{x_2^- - x_1^+}.
\end{aligned}$$

In solving this equation we have chosen D_{12} to be a pure phase. Note that the overall phase of the scattering matrix S_{12}^0 is not determined by this argument.

5.4 Bound State Spectrum

The one-loop dilatation operator in the $SU(2)$ sector is mapped onto the Hamiltonian of the Heisenberg spin chain. The scaling dimension's spectrum is determined by diagonalising the Heisenberg Hamiltonian. To be more explicit: at one-loop, the scaling dimension Δ of an operator is related to the energy of the corresponding eigenstate of the spin-chain as

$$\Delta = L + \frac{\lambda}{8\pi^2}E,$$

where $L = J_1 + J_2$, J_1 is the R-charge of the theory, and J_2 is the second conserved global charge. States of the spin chain with charges J_1 and J_2 have J_2 flipped spins in a periodic chain of length L, which is the number of sites in the spin chain. Eigenstates with a single flipped spin are known as magnons. Magnons have conserved energy ε and momentum p related by the dispersion relation

$$\varepsilon(p) = 4\sin^2\left(\frac{p}{2}\right).$$

Eigenstates in the sector with M flipped spins are formed as linear superpositions of M magnons. They are characterised by M individually conserved momenta p_k for $k = 1, \dots, M$ and have total energy

$$E = \sum_{k=1}^{M} \varepsilon(p_k) = \sum_{k=1}^{M} 4\sin^2\left(\frac{p_k}{2}\right).$$

Determining the energy levels is reduced to determining the allowed values of the momenta p_k. These are determined by the Bethe ansatz equations

$$e^{iLp_k} = \prod_{j\neq k} \mathcal{S}(p_k, p_j), \qquad \sum_{k=1}^{M} p_k = 0, \tag{15}$$

for $k = 1, \dots, m$. In this case $\mathcal{S}$ is the two-particle S-matrix which is given as

$$\mathcal{S}(p_k, p_j) = \frac{\varphi(p_k) - \varphi(p_j) + i}{\varphi(p_k) - \varphi(p_j) - i} \tag{16}$$

in terms of the phase function $\varphi(p) = \frac{1}{2}\cot\left(\frac{p}{2}\right)$. Note that the cot-function has the identity

$$\cot z = i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}.$$

The form in (16) of the two-particle S-matrix, with the use of the identity above, originates from [21]

$$\mathcal{S}(p_1, p_2) = -\frac{e^{i(p_1+p_2)} - 2e^{ip_1} + 1}{e^{i(p_1+p_2)} - 2e^{ip_2} + 1}, \tag{17}$$

which is the same two-body S-matrix from equation (15) [21]. This is a rewriting of A_{12} given at the end of section 5.3 with the use of the parameters $x^{\pm}$ described in (13). The limit $L \to \infty$ is taken, with M fixed, along with the momenta p_k of individual magnons also being held fixed. This limit is the standard thermodynamic limit of the spin chain. We refer to this limit as the Hofman and Maldacena (HM) limit. The main feature of the HM limit is that, as the chain becomes very long, the magnons become dilute. For the Heisenberg spin chain, the spectrum of magnon bound states in the thermodynamic limit is well-known. There is a single bound state of Q magnons, for each positive $Q \leq L/2$ with dispersion relation

$$\varepsilon_Q(p) = \frac{4}{Q} \sin^2\left(\frac{p}{2}\right).$$

Taking $L \to \infty$, this is effectively an infinite tower.

The recipe for finding these bound states is quite simple: two magnon bound states correspond to poles in the two-body S-matrix $\mathcal{S}(p_k, p_j)$. In particular, a pole is found in $\mathcal{S}(p_1, p_2)$ when

$$\varphi(p_1) - \varphi(p_2) = \frac{1}{2} \cot\left(\frac{p_1}{2}\right) - \frac{1}{2} \cot\left(\frac{p_2}{2}\right) = i,$$

which corresponds to a bound state with $U(1)$ charge $J_2 = Q = 2$ and momentum $p = p_1 + p_2$. These conditions are solved by setting

$$p_1 = \frac{p}{2} + iv \qquad p_2 = \frac{p}{2} - iv$$

in the above equation which yields $\cos\left(\frac{p}{2}\right) = e^v$. This yields a state with energy

$$E = \varepsilon(p_1) + \varepsilon(p_2) = 4\sin^2\left(\frac{p}{4} + i\frac{v}{2}\right) + 4\sin^2\left(\frac{p}{4} - i\frac{v}{2}\right) = 2\sin^2\left(\frac{p}{2}\right) = \varepsilon_2(p).$$

Thus, the position of the pole uniquely fixes the dispersion relation of the bound state.

The existence of the higher bound states with $Q > 2$, and their dispersion relation $\varepsilon_Q(p)$, can be inferred from singularities in the multi-particle S-matrix. For any integrable chain, this is given by a product of two-body factors. The corresponding pole appears when the momenta of the Q constituent magnons satisfy

$$\varepsilon(p_j) - \varepsilon(p_{j+1}) = i$$

for $j = 1, 2, \ldots, Q-1$. This condition is easily solved and leads directly to the bound state dispersion relation $\varepsilon_Q(p)$. Our previous expression for the S-matrix left the overall phase unspecified. The exact S-matrix is [18]

$$\mathcal{S}(p_k, p_j) = \frac{\varphi(p_k) - \varphi(p_j) + i}{\varphi(p_k) - \varphi(p_j) - i} \times \mathcal{S}_D(p_k, p_j),$$

where the phase function $\varphi(p)$ is now corrected to

$$\varphi(p) = \frac{1}{2} \cot\left(\frac{p}{2}\right) \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}.$$

The factor $\mathcal{S}_D$ is a "dressing factor" which is the undetermined overall phase of the full S-matrix. The dressing factor does not cancel the S-matrix pole which appears in $\mathcal{S}(p_k, p_j)$ when $\varphi(p_k) - \varphi(p_j) = i$. A question that comes up is what happens to the Q-magnon bound states and their dispersion law $\varepsilon_Q(p)$ described above when we move away from weak coupling? A proposal was made [4] that these states (bound states and dispersion law) survive for all values of the coupling and have the exact dispersion relation

$$\varepsilon_Q(p) = \frac{8\pi^2}{\lambda}\left[\sqrt{Q^2 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)} - Q\right].$$

This formula reduces to the dispersion relation $\varepsilon_Q(p) = \frac{4}{Q}\sin^2\left(\frac{p}{2}\right)$ of the Heisenberg spin chain at weak coupling. By setting $Q = 1$ one obtains the exact magnon dispersion relation $\varepsilon(p) = \frac{8\pi^2}{\lambda}\left[\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)} - 1\right]$. For $Q = 2$, the proposed bound state should correspond to the pole in the exact two-body S-matrix $\mathcal{S}(p_k, p_j) = \frac{\varphi(p_k)-\varphi(p_j)+i}{\varphi(p_k)-\varphi(p_j)-i} \times \mathcal{S}_D(p_k, p_j)$. The pole position should determine the dispersion relation exactly. It will now be explicitly verified.

For magnon momenta p_1 and p_2, the new pole condition reads

$$\frac{1}{2}\cot\left(\frac{p_1}{2}\right)\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p_1}{2}\right)} - \frac{1}{2}\cot\left(\frac{p_2}{2}\right)\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p_2}{2}\right)} = i$$

where we again set

$$p_1 = \frac{p}{2} + iv \qquad p_2 = \frac{p}{2} - iv$$

and solve for the bound state momentum $p = p_1 + p_2$ as a function of v. After some computation, we obtain a sixth-order polynomial equation, $P_6(t) = 0$, [4] in $t = \cos(p/2)$ with coefficients polynomial in e^v and $a = \lambda/4\pi^2$. The polynomial $P_6(t)$ can be factored exactly into the product of a quadratic $P_2(t)$ and a quartic $P_4(t)$ which are given as [4]

$$\begin{aligned}P_2(t) &= a(e^{2v} - 1)^2(1 + e^{2v} - 2e^v t)^2 - 4e^{2v}(1 + 6e^{2v} + e^{4v} - 4e^v t - 4e^{3v} t),\\ P_4(t) &= a(1 + e^{2v} - 2e^v t)^2(t^2 - 1) + 4e^v(t + e^{2v} t - e^v(1 + t^2)).\end{aligned}$$

The physical root is singled out by its weak-coupling behaviour $t = e^v$ needed for agreement with the corresponding formula for the Heisenberg spin chain discussed above. Taking the limit $a \to 0$, one may check with ease that the physical root belongs to the quartic equation $P_4(t) = 0$ rather than the quadratic. The next step is to extract the physical root of the quartic $P_4(t) = 0$, use it to eliminate v in the energy formula,

$$\begin{aligned}\varepsilon_2(p) &= \varepsilon(p_1) + \varepsilon(p_2)\\ &= \frac{8\pi^2}{\lambda}\left[\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{4} + i\frac{v}{2}\right)} + \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{4} - i\frac{v}{2}\right)}\right]\end{aligned}$$

and compare with the predicted dispersion relation $\varepsilon_Q(p)$ for the $Q = 2$ case. A necessary and sufficient condition for agreement with $\varepsilon_Q(p)$ is that the physical root of the quartic should also obey the corresponding energy conservation equation

$$\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{4} + i\frac{v}{2}\right)} + \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{4} - i\frac{v}{2}\right)} = \sqrt{4 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)}.$$

Squaring this equation twice and rewriting it in terms of $t = \cos(p/2)$, e^v and $a = \lambda/4\pi^2$, the same quartic equation $P_4(t) = 0$ is obtained [4], and we are done.

5.5 Conclusion

It was made explicitly clear why we associate a pole with a bound state as it is an important result that was required throughout this section. Following this, we reviewed work on magnons and the algebra of the scattering matrix; this determines the scattering matrix which determines the exact bound state spectrum. In determining the bound states we had to look at poles of the S-matrix. We will now explore the possibility that magnon bound states reproduce the spectrum of modes we found in section 3 from the expansion of $\mathcal{N} = 4$ SYM on S^3.

6 Comparisons of Obtained Spectra

6.1 Introduction

The goal of this section is to see if the spectrum of the bound states found in the previous section matches the spectrum from the harmonic expansion on the 3-sphere, S^3, constructed in section 3. We first specify what the relevant $\mathcal{N} = 4$ SYM symmetries are and we specify what the quantum numbers are for the fields we consider in the theory. From there we will introduce the notation for the Young diagram for supergroups and carefully spell out how to interpret the irreps associated to the newly constructed Young diagrams. Following from this, we identify the irreps that will describe the elementary magnon. We will finally test our hypothesis that magnon bound states reproduce the $L = 2$ mode on the 3-sphere. This must be continued to $L > 2$ to confirm our hypothesis.

6.2 Determining the Bound State spectrum

Our operator is built mainly from Z's - this is the field that builds the giant's worldvolume. The symmetries of $\mathcal{N} = 4$ SYM include:

- $SO(6)$ - this group rotates fields ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5, and ϕ_6 which are the $\mathcal{R}$-symmetry.
- $SO(4)$ - this group is from $SO(2,4)$ conformal symmetry and it is often called the conformal spin.

The field $Z_{(0)} = \phi_1 + i\phi_2$ is left invariant by $SO(4) \subset SO(6)$ and by $SO(4) \subset SO(4,2)$. Note that the term $Z_{(0)}$ is the S-wave of Z when we do the harmonic expansion on S^3. In the dual string theory,

- $SO(6) =$ isometry of S^5 and
- $SO(2,4) =$ isometry of AdS_5,

so we write

- $SO(4)_{S^5}$ for $SO(4) \subset SO(6)$ and
- $SO(4)_{AdS_5}$ for $SO(4) \subset SO(2,4)$.

Decompose each $SO(4)$ using $SO(4) \simeq SU(2)_L \times SU(2)_R$. The fields of $\mathcal{N} = 4$ SYM and their quantum numbers are:

Field	$SU(2)_{L,S^5}$	$SU(2)_{L,AdS_5}$	$SU(2)_{R,S^5}$	$SU(2)_{R,AdS_5}$
Z	1	1	1	1
$\bar{Z}$	1	1	1	1
ϕ_i	$\square$	1	$\square$	1
D_μ	1	$\square$	1	$\square$
$\psi_{\alpha\beta}$	$\square$	$\square$	1	1
$\psi_{\dot{\alpha}\dot{\beta}}$	1	1	$\square$	$\square$

This is one irreducible representation of $(PSU(2,2) \times PSU(2,2) \ltimes \mathbb{R}^3)$, the bifundamental representation. We have the following states in the bifundamental rep

$$|\phi^a\rangle \otimes |\phi^{\dot{a}}\rangle,\ |\psi^\alpha\rangle \otimes |\psi^{\dot{\alpha}}\rangle,\ |\phi^a\rangle \otimes |\psi^{\dot{\alpha}}\rangle,\ |\psi^\alpha\rangle \otimes |\phi^{\dot{a}}\rangle.$$

We write this using Young diagrams for supergroups:

$$(\boxslash, \boxslash) = (\square, 1; \square, 1) \oplus (\square, 1; 1, \square) \oplus (1, \square; \square, 1) \oplus (1, \square; 1, \square) \tag{18}$$

which is the quantum number of the $SU(2)^4$ subgroup. In the notation above, the semicolon separates the two $PSU(2,2)$ group reps; a left $(PSU(2,2)_L)$ and right $(PSU(2,2)_R)$. In each $PSU(2,2)$ there are two irreps: one representing $SU(2)$ for S^5 and one representing $SU(2)$ for AdS_5. The fields can be represented as follows:

$$\begin{aligned}
\phi_i &= (\square, 1; \square, 1)\\
D_\mu &= (1, \square; 1, \square)\\
\psi_{a\beta} &= (\square, \square; 1, 1)\\
\psi_{\dot{a}\dot{\beta}} &= (1, 1; \square, \square).
\end{aligned} \tag{19}$$

Thus, the elementary magnon is in the bifundamental of $PSU(2,2) \times PSU(2,2) \ltimes \mathbb{R}^3$.

The question to ask now is, what are the quantum numbers for a bound state of 2 magnons? The allowed quantum numbers are in the tensor product

$$(\boxslash, \boxslash) \otimes (\boxslash, \boxslash).$$

We again want the magnon bound state to be in a short rep of $(PSU(2,2) \times PSU(2,2) \ltimes \mathbb{R}^3)$ which implies [20]

$$C_1^2 - P_1 K_1 = \frac{1}{4} \qquad C_2^2 - P_2 K_2 = \frac{1}{4}$$

and

$$\begin{aligned}
(C_1 + C_2)^2 - (P_1 + P_2)(K_1 + K_2) &= 1\\
\Rightarrow 2C_1 C_2 - P_1 K_2 - P_2 K_1 &= \frac{1}{2}.
\end{aligned}$$

In this case: $\boxslash \otimes \boxslash = \boxslash\boxslash \oplus \begin{smallmatrix}\boxslash\\ \boxslash\end{smallmatrix}$, where

$$\begin{aligned}
\boxslash\boxslash &= (\square\square, 1) \oplus (\square, \square) \oplus \left(1, \begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)\\
\begin{smallmatrix}\boxslash\\ \boxslash\end{smallmatrix} &= \left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}, 1\right) \oplus (\square, \square) \oplus (1, \square\square).
\end{aligned}$$

Thus,

$$(\boxslash; \boxslash) \otimes (\boxslash; \boxslash) = (\boxslash\boxslash; \boxslash\boxslash) \oplus \left(\boxslash\boxslash; \begin{smallmatrix}\boxslash\\ \boxslash\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}\boxslash\\ \boxslash\end{smallmatrix}; \boxslash\boxslash\right) \oplus \left(\begin{smallmatrix}\boxslash\\ \boxslash\end{smallmatrix}; \begin{smallmatrix}\boxslash\\ \boxslash\end{smallmatrix}\right). \tag{20}$$

Consider each irrep on the RHS of equation (20). Using the explicit description (18) and the usual rules for multiplying Young diagrams, we find

- $(\boxslash\boxslash, \boxslash\boxslash)$:

$$
\begin{aligned}
(\boxslash\boxslash, \boxslash\boxslash) =& (\square\square, 1; \square\square, 1) \oplus (\square\square, 1; \square, \square) \oplus (\square\square, 1; 1, \substack{\square\\\square}) \oplus (\square, \square; \square\square, 1) \oplus (\square, \square; \square, \square) \oplus \\
& \oplus (\square, \square; 1, \substack{\square\\\square}) \oplus (1, \substack{\square\\\square}; \square\square, 1) \oplus (1, \substack{\square\\\square}; \square, \square) \oplus (1, \substack{\square\\\square}; 1, \substack{\square\\\square}).
\end{aligned} \tag{21}
$$

- $(\boxslash\boxslash, \substack{\boxslash\\\boxslash})$:

$$
\begin{aligned}
(\boxslash\boxslash, \substack{\boxslash\\\boxslash}) =& (\square\square, 1; \substack{\square\\\square}, 1) \oplus (\square\square, 1; \square, \square) \oplus (\square\square, 1; 1, \square\square) \oplus (\square, \square; \substack{\square\\\square}, 1) \oplus (\square, \square; \square, \square) \oplus \\
& \oplus (\square, \square; 1, \square\square) \oplus (1, \substack{\square\\\square}; \substack{\square\\\square}, 1) \oplus (1, \substack{\square\\\square}; \square, \square) \oplus (1, \substack{\square\\\square}; 1, \square\square).
\end{aligned} \tag{22}
$$

- $(\substack{\boxslash\\\boxslash}, \substack{\boxslash\\\boxslash})$:

$$
\begin{aligned}
(\substack{\boxslash\\\boxslash}, \substack{\boxslash\\\boxslash}) =& (\substack{\square\\\square}, 1; \substack{\square\\\square}, 1) \oplus (\substack{\square\\\square}, 1; \square, \square) \oplus (\substack{\square\\\square}, 1; 1, \square\square) \oplus (\square, \square; \substack{\square\\\square}, 1) \oplus (\square, \square; \square, \square) \oplus \\
& \oplus (\square, \square; 1, \square\square) \oplus (1, \square\square; \substack{\square\\\square}, 1) \oplus (1, \square\square; \square, \square) \oplus (1, \square\square; 1, \square\square).
\end{aligned} \tag{23}
$$

Note that irrep $(\substack{\boxslash\\\boxslash}, \boxslash\boxslash)$ is obtained by swapping $L \leftrightarrow R$ in equation (22). This shows that these are 3 possible independent bound states. Now, the mode with $L = 2$ on S^3 is $(\square\square, 1; \square\square, 1)$. This implies that only equation (21) is a parton with momentum 2. If the bound states are to reproduce the correct spectrum of harmonics, there should be no bound states in representations 22 and 23. This is a very definite prediction, but is it true? Inspecting equation (19), note that

$$
\begin{aligned}
\phi_i \otimes \phi_j &= (\square\square, 1; \square\square, 1) \oplus (\square\square, 1; 1, 1) \oplus (1, 1; \square\square, 1) \oplus (1, 1; 1, 1), \\
D_a \otimes D_b &= (1, \square\square; 1, \square\square) \oplus (1, \square\square; 1, 1) \oplus (1, 1; 1, \square\square) \oplus (1, 1; 1, 1), \\
D_a \otimes \phi_i &= (\square, \square; \square, \square) = \psi_{\dot{a}\dot{\beta}} \otimes \psi_{a\beta}, \\
\psi_{a\beta} \otimes \psi_{b\alpha} &= (\square\square, 1; 1, \square\square) \oplus (\square\square, 1; 1, 1) \oplus (1, 1; 1, \square\square) \oplus (1, 1; 1, 1), \\
\psi_{\dot{a}\dot{\beta}} \otimes \psi_{\dot{b}\dot{\alpha}} &= (1, \square\square; \square\square, 1) \oplus (1, \square\square; 1, 1) \oplus (1, 1; \square\square, 1) \oplus (1, 1; 1, 1), \\
\phi_i \otimes \psi_{a\beta} &= (1, 1; \square, \square) \oplus (\square\square, 1; \square, \square), \\
\phi_i \otimes \psi_{\dot{a}\dot{\beta}} &= (\square, \square; 1, 1) \oplus (\square, \square; \square\square, 1), \\
D_a \otimes \psi_{a\beta} &= (\square, \square; 1, 1) \oplus (\square, \square; 1, \square\square), \\
D_a \otimes \psi_{\dot{a}\dot{\beta}} &= (1, 1; \square, \square) \oplus (1, \square\square; \square, \square).
\end{aligned}
$$

From the above, we see the following:

- $$(\boxslash\boxslash \otimes \boxslash\boxslash) = (\phi_i \otimes \phi_j) \oplus (\phi_i \otimes \psi_{a\beta}) \oplus (\phi_i \otimes \psi_{\dot{a}\dot{\beta}}) \oplus (\psi_{\dot{a}\dot{\beta}} \otimes \psi_{a\beta}) = (21),$$

  which is the bound state Dorey [4] found, which we discussed in section 5.

- $$(\boxslash\boxslash, \substack{\boxslash\\\boxslash}) = (\psi_{a\beta} \otimes \psi_{b\alpha}) \oplus (\phi_i \otimes \psi_{a\beta}) \oplus (D_a \otimes \psi_{a\beta}) \oplus (D_a, \phi_i) = (22).$$

  There is no $(\psi \otimes \psi)$ bound state so (22) does not lead to a bound state - see [22].

- $$\left(\begin{array}{|c|}\hline / \\ \hline / \\ \hline\end{array}, \begin{array}{|c|}\hline / \\ \hline / \\ \hline\end{array}\right) = (D_a \otimes D_b) \oplus (D_a \otimes \psi_{\dot{a}\dot{\beta}}) \oplus (D_a \otimes \psi_{a\beta}) \oplus (\psi_{\dot{a}\dot{\beta}} \otimes \psi_{a\beta}) = (23).$$

  There is no $(D_a \otimes D_b)$ bound state so (23) is also not an allowed bound state - see [22].

This implies that our idea has passed a highly non-trivial check! The only two magnon bound state perfectly matched the $L = 2$ mode (which corresponds to the $(1, 1)$ irrep seen in section 3.5) in the expansion of $\mathcal{N} = 4$ SYM on S^3.

6.3 Conclusion

By tabulating the quantum numbers of the fields of $\mathcal{N} = 4$ SYM theory appropriately, we could nicely construct the Young diagram of supergroup irreps. The irrep spells out clearly how one reads off the associated left and right $PSU(2, 2)$ quantum numbers of fields. By taking the tensor product of two copies of the magnon representation, we obtained four possible irreps for the bound state of two magnons. Of the four possible representations, only one matches the spectrum obtained from $\mathcal{N} = 4$ SYM on S^3. A detailed analysis of the bound state spectrum [4],[22] then revealed that only the irrep matching $\mathcal{N} = 4$ SYM on S^3 is in fact realised. This is highly non-trivial evidence for our hypothesis.

7 Conclusions and Future Endeavours

In this dissertation the question we set out to tackle was if there is an emergent Yang-Mills theory coming from the low energy description of branes and open strings. The question is highly non-trivial because this new Yang-Mills theory had, and still has, no connection to the original gauge symmetry of the CFT. This question requires that we explore a large N but non-planar limit of the theory, and to do this we used new methods developed in group representation theory. We studied the dilatation operator D in $\mathcal{N} = 4$ SYM theory since its eigenvalue, the anomalous dimension, is mapped to the energy of the open string in the IIB string theory. We set out to construct the spherical harmonics from the harmonic expansion on the 3-sphere, S^3, to understand the theory of the giant graviton's worldvolume. The light-front parton picture was examined from work done in [3] since it explained how one can "glue" single momentum modes together to obtain higher momentum modes, and we believe that this procedure is described dynamically using magnon bound states [4]. Following from this, we worked on determining the exact magnon bound state spectrum. We ended off by testing our hypothesis and seeing if the spectrum of the bound states matched the harmonic spectrum from the harmonic expansion on the 3-sphere, S^3. We have given a non-trivial check that the bound state spectrum does match the spectrum coming from $\mathcal{N} = 4$ SYM.

The reason why this research is important is because the appearance of an emergent Yang-Mills theory allows us to understand, in greater detail, the gravity/gauge duality in the realm of branes and their excitations at low energy. As a future project, we wish to construct this new Yang-Mills theory and we will start off by trying to match the spectra of magnon bound states and mode expansions on S^3 in general. Much remains to be done.

A Schur's Lemmas and the Fundamental Orthogonality Relation

A.1 Schur's Lemmas

A.1.1 Lemma 1:

If we have some matrix A obeying $\Gamma_R(g)A = A\Gamma_R(g)$, $\forall g \in \mathcal{G}$, and R is an irreducible representation then

$$A = \lambda \cdot \mathbb{1},$$

where λ is some constant.

Proof: Imagine that all eigenvalues of A are zero. Then the lemma is true with $\lambda = 0$. Now consider the case that A has at least one non-zero eigenvalue. Then we have

$$A|v\rangle = \eta|v\rangle,$$

where η is the non-zero eigenvalue and $|v\rangle$ is the corresponding eigenvector. From that result, it follows that

$$\begin{gathered}\Gamma_R(g)A|v\rangle = \eta\Gamma_R(g)|v\rangle \\ \Rightarrow A\Gamma_R(g)|v\rangle = \eta\Gamma_R(g)|v\rangle.\end{gathered}$$

Denote $|v_g\rangle \equiv \Gamma_R(g)|v\rangle$ which also has eigenvalue η for *any* $g \in \mathcal{G}$. Since there are no invariant subspaces, it implies that

$$\begin{gathered}\Gamma_R(g)|v\rangle \equiv |v_g\rangle \text{ span the whole space} \\ \Rightarrow \text{Every vector in the space has eigenvalue } \eta \\ \Rightarrow A = \lambda \cdot \mathbb{1} \text{ with } \lambda = \eta \qquad \square\end{gathered}$$

A.1.2 Lemma 2:

Given *any* two inequivalent irreducible representations R,S the only solution to

$$\Gamma_R(g)A = A\Gamma_S(g) \qquad \forall g \in \mathcal{G}$$

is $A = 0$.

Proof: Denote the dimension of R by d_R and of S by d_S.

Case 1: $d_R = d_S$: Assume that A has no zero eigenvalues. Then A^{-1} exists, so that

$$\begin{gathered}\Gamma_R(g)AA^{-1} = A\Gamma_S(g)A^{-1} \\ \Rightarrow \Gamma_R(g) = A\Gamma_S(g)A^{-1},\end{gathered}$$

but this is a contradiction since R and S are inequivalent, thus it implies that

$$\begin{gathered}A \text{ has at least one zero eigenvalue} \\ \Rightarrow \text{There is a } |v\rangle \text{ so that } A|v\rangle = 0 \\ \Rightarrow \Gamma_R(g)A|v\rangle = 0 = A\Gamma_S(g)|v\rangle.\end{gathered}$$

Since S is an inequivalent irreducible representation, $\Gamma_S(g)|v\rangle$ span the whole space, which implies A annihilates every vector in space, which implies that $A = 0$.

Case 2: $d_R < d_S$: Suppose matrix A is of the form $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$, such that

$$A\vec{v} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \vec{A_1} \cdot \vec{v} \\ \vec{A_2} \cdot \vec{v} \end{bmatrix}.$$

It is always possible to find a $|v\rangle$ so that

$$\begin{aligned} A|v\rangle &= 0 \\ \Rightarrow \Gamma_R(g)A|v\rangle &= 0 = A\Gamma_S(g)|v\rangle. \end{aligned}$$

Since S is an irrep, then $\Gamma_S(g)|v\rangle$ span the whole space, which implies A annihilates every vector in space, which implies that $A = 0$.

Case 3: $d_R > d_S$: Suppose matrix A is of the form $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$, such that

$$A\vec{v} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \vec{w}.$$

Now in this case we still have $\Gamma_R(g)A|v\rangle = A\Gamma_S(g)|v\rangle$, but notice that on the left hand side of the equation there is a sum of 3 linearly independent vectors whereas on the right hand side of the equation there is a sum of 2 linearly independent vectors. This is a contradiction and the only resolution is if $A = 0$. $\square$

B Many Traces Into a Single Trace

Suppose we have

$$((1\ 2\ 3))^L_I (Z^{\otimes 3})^I_J ((1\ 3\ 2))^J_K = \delta^{l_1}_{i_2}\delta^{l_2}_{i_3}\delta^{l_3}_{i_1} Z^{i_1}_{j_1} Z^{i_2}_{j_2} Z^{i_3}_{j_3} \delta^{j_1}_{k_3}\delta^{j_2}_{k_1}\delta^{j_3}_{k_2} = Z^{l_3}_{k_3} Z^{l_1}_{k_1} Z^{l_2}_{k_2},$$

which essentially is the swapping of positions of $Z^{l_3}_{k_3}$, $Z^{l_1}_{k_1}$, and $Z^{l_2}_{k_2}$ according to the permutation $(1\ 3\ 2)$ read from right to left, i.e. 2 goes to 3, 3 goes to 1, and 1 goes to 2. In general, we have

$$(\sigma^{-1})^L_I (Z^{\otimes n})^I_J (\sigma)^J_K = (Z^{\otimes n})^L_K,$$

where positions of $Z^{\otimes n}$ are swapped according to σ being read from right to left. It is a symmetry to swap positions if we are dealing with bosons. If we are dealing with fermions, then when we swap positions we pick up an extra phase. Observables are given by taking a trace:

$$\begin{aligned} \mathrm{Tr}(\sigma Z^{\otimes n}) &= Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} \\ &= \mathrm{Tr}(\sigma\rho Z^{\otimes n}\rho^{-1}) \\ &= \mathrm{Tr}(\rho^{-1}\sigma\rho Z^{\otimes n}), \end{aligned}$$

where in the last line the cyclicity of the trace was used and ρ is a permutation that permutes the Z fields. Considering the observables $\mathrm{Tr}(Z^2)\,\mathrm{Tr}(Z)$, $\mathrm{Tr}(Z^3)$, and $\mathrm{Tr}(Z)^3$ which are 2 traces, 1 trace, and 3 traces respectively and they all live in the V_N. They can be rewritten as $\mathrm{Tr}((1\ 2)\cdot Z^{\otimes 3})$, $\mathrm{Tr}((1\ 2\ 3)\cdot Z^{\otimes 3})$, and $\mathrm{Tr}(\mathbb{1}\cdot Z^{\otimes 3})$ respectively which are all single trace expressions in $V_N^{\otimes 3}$. Thus we see that multitrace structures in V_N can be written as a single trace in the bigger space $V_N^{\otimes 3}$. This conclusion holds for any number of matrices: any multi-trace structure build using n Z's can be written as a single trace structure in $V_N^{\otimes n}$.

C Derivation of the delta function on a finite group

Consider a finite group G. There is a representation, called the regular representation $\Gamma_{rr}(g)$, that can be read from the multiplication table. To each group element, associate a basis vector $|g_i\rangle$ where $i = 1, \ldots, |G|$. The regular representation is then defined by

$$\Gamma_{rr}(g_1)|g_2\rangle = |g_1 \cdot g_2\rangle,$$

for $g_1, g_2 \in G$. In the multiplication table of S_n, each row has entries of zero except for a single entry which is one. The element that maps to the identity permutation is the identity group element. Since each column/row contains each element of the group exactly once, considering the definition of $\Gamma_{rr}(g)$, we see that it is only the matrix representing the identity that has any diagonal element non-zero. $\Gamma_{rr}(1)$ is the $|G|$ dimensional identity matrix. This implies that

$$\chi_{rr}(g) = \mathrm{Tr}\left(\Gamma_{rr}(g)\right) = |G|\delta(g),$$

where $\delta(g)$ is 1 if g is the identity, and zero otherwise. It should be noted that the regular representation is reducible. By denoting the irreducible representations of G by r_q, we find that r_q appears

$$\begin{aligned}
n_q &= \frac{1}{|G|}\sum_{g\in G}\chi_{rr}(g)\chi_{r_q}(g^{-1}) \\
&= \frac{1}{|G|}\chi_{rr}(1)\chi_{r_q}(1) \\
&= d_{r_q}
\end{aligned}$$

times. The above computation has used

$$\chi_R(g) = \sum_a n_a \chi_{r_a}(g)$$

$$\sum_{g\in G}\chi_{r_a}(g)\chi_{r_q}(g^{-1}) = |G|\delta_{aq}$$

which is the character formula for characters in a reducible representation, and the character orthogonality relation respectively. Combining this result with what we had before, we see that

$$\chi_{rr} = \sum_{r_q} d_{r_q}\chi_{r_q}(g) = |G|\delta(g)$$

which can then be rearranged such that the delta function on the group G is

$$\delta(g) = \frac{1}{|G|}\sum_{r_q} d_{r_q}\chi_{r_q}(g).$$

D Projector $O_{ab}^{R,r}$

The projector which projects onto matrix elements of the irrep r is given as

$$O_{ab}^{R,r} = \frac{d_r}{|H|} \sum_{h \in H} \Gamma_{ab}^r(h^{-1}) \Gamma_{AB}^R(h).$$

As we have remarked before, we should refer to this as an intertwining operator. We wish to prove that this is an intertwiner by showing that this operator obeys the correct matrix algebra. Towards this end, consider

$$\begin{aligned} O_{ab}^{R,r} O_{bc}^{R,s} &= \frac{d_r}{|H|} \sum_{h \in H} \Gamma_{ab}^r(h^{-1}) \Gamma_{AB}^R(h) \frac{d_s}{|H|} \sum_{h' \in H} \Gamma_{bc}^s((h')^{-1}) \Gamma_{BC}^R(h') \\ &= \frac{d_r d_s}{|H|^2} \sum_{h,h' \in H} \Gamma_{ab}^r(h^{-1}) \Gamma_{bc}^s((h')^{-1}) \Gamma_{AB}^R(h) \Gamma_{BC}^R(h'). \end{aligned}$$

Since we sum over all elements of the group H we can replace h^{-1} with h. With the same argument, we do the same with h' and $(h')^{-1}$. Thus we now have,

$$\begin{aligned} O_{ab}^{R,r} O_{bc}^{R,s} &= \frac{d_r d_s}{|H|^2} \sum_{h,h' \in H} \Gamma_{ab}^r(h) \Gamma_{bc}^s(h') \Gamma_{AB}^R(h) \Gamma_{BC}^R(h') \\ &= \frac{d_r d_s}{|H|^2} \sum_{h,h' \in H} \Gamma_{ab}^r(h) \Gamma_{bc}^s(h') \Gamma_{AC}^R(hh'). \end{aligned}$$

Changing our group elements from h,h' to h,ψ where $\psi = hh'$ so then we can write $h' = h^{-1}\psi$. Our calculation now becomes

$$\begin{aligned} O_{ab}^{R,r} O_{bc}^{R,s} &= \frac{d_r d_s}{|H|^2} \sum_{h,\psi \in H} \Gamma_{ab}^r(h) \Gamma_{bc}^s(h^{-1}\psi) \Gamma_{AC}^R(\psi) \\ &= \frac{d_r d_s}{|H|^2} \sum_{\psi \in H} \left[\sum_{h \in H} \Gamma_{ab}^r(h) \Gamma_{bc}^s(h^{-1}\psi) \right] \Gamma_{AC}^R(\psi). \end{aligned}$$

Inspecting what we have in brackets:

$$\begin{aligned} \sum_{h \in H} \Gamma_{ab}^r(h) \Gamma_{bc}^s(h^{-1}\psi) &= \sum_{h \in H} \Gamma_{ab}^r(h) \Gamma_{bd}^s(h^{-1}) \Gamma_{dc}^s(\psi) \\ &= \left(\frac{|H|}{d_r} \delta_{rs} \delta_{ad} \delta_{bb} \right) \Gamma_{dc}^s(\psi) \\ &= \frac{|H|}{d_r} \delta_{rs} \Gamma_{ac}^s(\psi). \end{aligned}$$

Plugging this result back into our original calculation, we get

$$
\begin{aligned}
O^{R,r}_{ab} O^{R,s}_{bc} &= \frac{d_r d_s}{|H|^2} \sum_{\psi \in H} \left[\frac{|H|}{d_r} \delta_{rs} \Gamma^s_{ac}(\psi) \right] \Gamma^R_{AC}(\psi) \\
&= \frac{d_s}{|H|} \delta_{rs} \sum_{\psi \in H} \Gamma^s_{ac}(\psi) \Gamma^R_{AC}(\psi) \\
&= \frac{d_s}{|H|} \delta_{rs} \sum_{\psi \in H} \Gamma^s_{ac}(\psi^{-1}) \Gamma^R_{AC}(\psi) \\
&\equiv \delta_{rs} O^{R,s}_{ac},
\end{aligned}
$$

which is indeed the expected matrix algebra if r is equal to s. In the second last line, ψ was replaced with ψ^{-1} which is again justified since we are summing over the complete group H.

E Dirac bracket

The Dirac bracket we wish to prove is given as

$$
\{\varphi(x^-, x), \varphi(y^-, y)\}_{\mathrm{D}} = (\partial_-)\delta(x^- - y^-)\delta(x - y),
$$

where $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3)$, $x = (x^1, x^2)$, and $\partial_- = \frac{\partial}{\partial x^-}$. Recall that the Lagrangian density we consider is

$$
\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi),
$$

and in light-cone coordinates is given by

$$
\mathcal{L} = \partial_+ \varphi \partial_- \varphi - \frac{1}{2} \nabla \varphi \cdot \nabla \varphi - V(\varphi).
$$

The generalised momentum is computed to be

$$
\begin{aligned}
\Pi &= \frac{\partial \mathcal{L}}{\partial(\partial_+ \varphi)} = \partial_- \varphi \\
\Rightarrow & \Pi - \partial_- \varphi \equiv \chi_1 = 0,
\end{aligned}
$$

where χ_1 is a primary constraint. The Hamiltonian density is given by

$$
\begin{aligned}
\mathcal{H} = \Pi(\partial_+ \varphi) - \mathcal{L} &= \partial_+ \varphi \partial_- \varphi - \partial_+ \varphi \partial_- \varphi + \frac{1}{2} \nabla \varphi \cdot \nabla \varphi + V(\varphi) \\
&= \frac{1}{2} \nabla \varphi \cdot \nabla \varphi + V(\varphi),
\end{aligned}
$$

and thus the Hamiltonian is given by

$$
H = \int dx^- dx \; \mathcal{H} = \int dx^- dx \left[\frac{1}{2} \nabla \varphi \cdot \nabla \varphi + V(\varphi) \right].
$$

However, the Hamiltonian above does not contain velocities nor momenta. The primary constraint is added [23] on the Hamiltonian as follows

$$H_T = H + \int dx^- dx \; u(x^-, x)[\Pi - \partial_- \varphi],$$

where $u(x^-, x)$ is unknown and is to be determined. Now, since $\chi_1 = 0 \Rightarrow \dot{\chi}_1 = \partial_+ \chi_1 = \{\chi_1, H_T\} = 0$. Computing this Poisson bracket, we get

$$\begin{aligned}
\{\chi_1, H_T\} &= \Big\{\Pi(x^-, x) - \partial_- \varphi(x^-, x), \int dy^- dy \Big[\frac{1}{2}\nabla\varphi(y^-, y)\cdot\nabla\varphi(y^-, y) + V(\varphi(y^-, y)) + \\
&\quad + u(y^-, y)[\Pi(y^-, y) - \partial_-\varphi(y^-, y)]\Big]\Big\} \\
&= \int dy^- dy \Big\{\Pi(x^-, x), \Big(\frac{1}{2}\nabla\varphi(y^-, y)\cdot\nabla\varphi(y^-, y) - u(y^-, y)\partial_-\varphi(y^-, y)\Big)\Big\} - \\
&\quad - \int dy^- dy \left\{\partial_-\varphi(x^-, x), u(y^-, y)\Pi(y^-, y)\right\} \\
&= \int dy^- dy \Big[\delta(x^- - y^-)\delta(x - y)\nabla^2\varphi(y^-, y) - \delta(x^- - y^-)\delta(x - y)\partial_- u(y^-, y) - \\
&\quad - V'(\varphi(y^-, y))\delta(x^- - y^-)\delta(x - y) - \partial_- u(y^-, y)\delta(x^- - y^-)\delta(x - y)\Big] \\
&= \nabla^2\varphi(x^-, x) - 2\partial_- u(x^-, x) - V'(\varphi(x^-, x)) \\
\Rightarrow \partial_- u(x^-, x) &= \frac{1}{2}\nabla^2\varphi(x^-, x) - \frac{1}{2}V'(\varphi(x^-, x)),
\end{aligned}$$

where $V'(\varphi)$ is the first derivative of the potential with respect to φ. In the calculation above, the following was used

$$\begin{aligned}
\frac{\delta}{\delta\varphi(y)}(\nabla\varphi(x)\cdot\nabla\varphi(x)) &= 2\nabla\delta(x - y)\cdot\nabla\varphi(x) \\
\Rightarrow \int dy \frac{\delta}{\delta\varphi(y)}(\nabla\varphi(x)\cdot\nabla\varphi(x)) &= \int dy\Big(2\big[\nabla\cdot(\delta(x - y)\nabla\varphi(x)) - \delta(x - y)\nabla\cdot\nabla\varphi(x)\big]\Big) \\
&= \int dy(-2\delta(x - y)\nabla\cdot\nabla\varphi(x)),
\end{aligned}$$

since boundary terms vanish under integration. Another fact used is

$$\begin{aligned}
-u(y^-, y)\partial_-\varphi(y^-, y) &= -\partial_-[u(y^-, y)\varphi(y^-, y)] + \varphi(y^-, y)\partial_- u(y^-, y) \\
\Rightarrow \int dy^- dy[u(y^-, y)\partial_-\varphi(y^-, y)] &= \int dy^- dy\left[-\partial_-[u(y^-, y)\varphi(y^-, y)] + \varphi(y^-, y)\partial_- u(y^-, y)\right] \\
&= \int dy^- dy\left[\varphi(y^-, y)\partial_- u(y^-, y)\right],
\end{aligned}$$

and once again, boundary terms vanish under integration. The final fact that was used is

$$\begin{aligned}
\{\varphi(x^-, x), \varphi(y^-, y)\} &= 0, \\
\{\Pi(x^-, x), \Pi(y^-, y)\} &= 0.
\end{aligned}$$

By computing $\partial_+\varphi = \{\varphi, H\}$, we get [23]

$$\begin{aligned}
\partial_+\varphi(x^-,x) = \{\varphi(x^-,x), H(y^-,y)\} &= \int dy^- dy \{\varphi(x^-,x), u(y^-,y)\Pi(y^-,y)\} \\
&= \int dy^- dy \delta(x^- - y^-)\delta(x-y)u(y^-,y) \\
&= u(x^-,x).
\end{aligned}$$

Substituting this result into the previous one, we find

$$\begin{aligned}
\partial_- u(x^-,x) = \partial_-\partial_+\varphi = \frac{1}{2}\nabla^2\varphi(x^-,x) - \frac{1}{2}V'(\varphi(x^-,x)) \\
\Rightarrow 2\partial_-\partial_+\varphi - \nabla^2\varphi(x^-,x) + V'(\varphi(x^-,x)) = 0 \\
\Rightarrow \chi_2 \equiv \Box\varphi(x^-,x) + V'(\varphi(x^-,x)) = 0,
\end{aligned}$$

which is our one and only secondary constraint. Therefore, the Dirac bracket is of the form [23]

$$\begin{aligned}
\{A(x^-,x), B(y^-,y)\}_{\mathrm{D}} &= \{A(x^-,x), B(y^-,y)\}_{\mathrm{D}} \\
&- \int du^- du \int dv^- dv \{A(x^-,x), \chi(u^-,u)\} M^{-1}(u^-,u,v^-,v)\{\chi(v^-,v), B(y^-,y)\},
\end{aligned}$$

where $M(x,y) = \{\chi(x), \chi(y)\}$ which is similar to $M_{ij} = \{\chi_i, \chi_j\}$ [23]. We only need to compute one component since $M_{ii} = 0$ and $M_{ij} = -M_{ji}$ for $i \neq j$. The component to work out is M_{12} and it is given by

$$\begin{aligned}
M_{12} = \{\chi_i, \chi_j\} &= \Big\{\big(\Pi(x^-,x) - \partial_-\varphi(x^-,x)\big), \big(\Box\varphi(y^-,y) + V'(\varphi(y^-,y))\big)\Big\} \\
&= \Big\{\Pi(x^-,x), \big(\Box\varphi(y^-,y) + V'(\varphi(y^-,y))\big)\Big\} - \Big\{\partial_-\varphi(x^-,x), \big(\Box\varphi(y^-,y) + V'(\varphi(y^-,y))\big)\Big\} \\
&= \Big\{\Pi(x^-,x), \big(\Box\varphi(y^-,y) + V'(\varphi(y^-,y))\big)\Big\} - 0 \\
&= -\Box[\delta(x^- - y^-)\delta(x-y)] - \delta(x^- - y^-)\delta(x-y)V''(\varphi(y^-,y)) \\
&= -\left[\Box[\delta(x^- - y^-)\delta(x-y)] + \delta(x^- - y^-)\delta(x-y)V''(\varphi(y^-,y))\right] \\
\Rightarrow (M_{12})^{-1} &= \left[\Box[\delta(x^- - y^-)\delta(x-y)] + \delta(x^- - y^-)\delta(x-y)V''(\varphi(y^-,y))\right]^{-1}.
\end{aligned}$$

The reason we have the inverse as shown above is due to the fact that if we have a matrix of the form

$$M_{ij} = \begin{bmatrix} 0 & -A \\ A & 0 \end{bmatrix},$$

then its inverse is

$$(M^{-1})_{ij} = \begin{bmatrix} 0 & \frac{1}{A} \\ -\frac{1}{A} & 0 \end{bmatrix},$$

where in our case $A = \Box[\delta(x^- - y^-)\delta(x-y)] + \delta(x^- - y^-)\delta(x-y)V''(\varphi(y^-, y))$. Now, the Dirac bracket between the field φ and the generalised momentum Π is given by

$$\begin{aligned}
\{\varphi(x^-, x), \Pi(y^-, y)\}_D &= \{\varphi(x^-, x), \Pi(y^-, y)\} - \int du^- du \int dv^- dv \{\varphi(x^-, x), \chi(u^-, u)\}\times \\
&\times \left[\Box[\delta(u^- - v^-)\delta(u-v)] + \delta(u^- - v^-)\delta(u-v)V''(\varphi(v^-, v))\right]^{-1} \{\chi(v^-, v), \Pi(y^-, y)\} \\
&= \delta(x^- - y^-)\delta(x-y) - \int du^- du \int dv^- dv \{\varphi(x^-, x), (\Pi(u^-, u) - \partial_-\varphi(u^-, u))\}\times \\
&\times \left[\Box[\delta(u^- - v^-)\delta(u-v)] + \delta(u^- - v^-)\delta(u-v)V''(\varphi(v^-, v))\right]^{-1} \times \\
&\times \{(\Box\varphi(v^-, v) + V'(\varphi(v^-, v))), \Pi(y^-, y)\} \\
&= \delta(x^- - y^-)\delta(x-y) - \int du^- du \int dv^- dv [\delta(x^- - u^-)\delta(x-u)]\times \\
&\times \left[\Box[\delta(u^- - v^-)\delta(u-v)] + \delta(u^- - v^-)\delta(u-v)V''(\varphi(v^-, v))\right]^{-1} \times \\
&\times \left[\Box[\delta(u^- - v^-)\delta(u-v)] + \delta(u^- - v^-)\delta(u-v)V''(\varphi(v^-, v))\right] \\
&= \delta(x^- - y^-)\delta(x-y) + \int dv^- dv\times \\
&\times \left[\Box[\delta(x^- - v^-)\delta(x-v)] + \delta(x^- - v^-)\delta(x-v)V''(\varphi(v^-, v))\right]^{-1} \times \\
&\times \left[-\Box[\delta(u^- - v^-)\delta(u-v)] + \delta(u^- - v^-)\delta(u-v)V''(\varphi(v^-, v))\right].
\end{aligned}$$

Note that [23]

$$\int dz^- dz M^{-1}(x, z) M(z, y) = \delta(x^- - y^-)\delta(x-y).$$

Thus, the Dirac bracket becomes

$$\begin{aligned}
\{\varphi(x^-, x), \Pi(y^-, y)\}_D &= \delta(x^- - y^-)\delta(x-y) + \int dv^- dv\times \\
&\times \left[\Box[\delta(x^- - v^-)\delta(x-v)] + \delta(x^- - v^-)\delta(x-v)V''(\varphi(v^-, v))\right]^{-1} \times \\
&\times \left[-\Box[\delta(u^- - v^-)\delta(u-v)] + \delta(u^- - v^-)\delta(u-v)V''(\varphi(v^-, v))\right] \\
&= \delta(x^- - y^-)\delta(x-y) + \delta(x^- - y^-)\delta(x-y) \\
&= 2\delta(x^- - y^-)\delta(x-y) \\
\Rightarrow \mathcal{N}\{\varphi(x^-, x), \Pi(y^-, y)\}_D &= \delta(x^- - y^-)\delta(x-y).
\end{aligned}$$

Recall that $\Pi = \partial_-\varphi$, thus [23]

$$\begin{aligned}
\mathcal{N}\{\varphi(x^-, x), \Pi(y^-, y)\}_D = \mathcal{N}\{\varphi(x^-, x), \partial_-\varphi(y^-, y)\}_D &= \delta(x^- - y^-)\delta(x-y) \\
\Rightarrow \mathcal{N}\{\varphi(x^-, x), \varphi(y^-, y)\}_D &= (\partial_-)^{-1}\delta(x^- - y^-)\delta(x-y),
\end{aligned}$$

thus completing the proof $\Box$.

References

[1] R. Bhattacharyya, S. Collins and R. de Mello Koch, "Exact Multi-Matrix Correlators,"JHEP 0803, 044 (2008) [arXiv:0801.2061 [hep-th]].

[2] R. Bhattacharyya, R. de Mello Koch and M. Stephanou, "Exact Multi-Restricted Schur Polynomial Correlators," JHEP 0806, 101 (2008) [arXiv:0805.3025 [hep-th]].

[3] A. Jevicki, "Light-Front Partons and Dimensional Reduction in Relativistic Field Theory," Phys. Rev. D57, 5955-5958 (1998) [hep-th/9712088].

[4] N. Dorey, "Magnon Bound States and the AdS/CFT Correspondence," J. Phys. A39, 13119-13128 (2006) [hep-th/0604175].

[5] V. Balasubramanian, D. Berenstein, B. Feng and M. Huang, "D-branes in Yang-Mills theory and Emergent Gauge Symmetry," JHEP 006, 03 (2005) [arXiv: 0411.205 [hep-th]].

[6] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," Nucl. Phys. B 72, 461 (1974).

[7] E. Witten, "Baryons in the 1/n Expansion," Nucl. Phys. B160, 57 (1979).

[8] S. Corley, A. Jevicki and S. Ramgoolam, "Exact correlators of giant gravitons from dual N=4 SYM theory," Adv.Theor.Math.Phys. 5, 809 (2002) [hep-th/0111222].

[9] S. Corley and S. Ramgoolam, "Finite factorization equations and sum rules for BPS correlators in N=4 SYM theory," Nucl. Phys. B641, 131 (2002) [hep-th/0205221].

[10] R. de Mello Koch, M. Dessein, D. Giataganas, and C. Mathwin, "Giant Graviton Oscillators," JHEP B 10, 009 (2011).

[11] R. de Mello Koch, S. Ramgoolam, "A double coset ansatz for integrability in AdS/CFT," JHEP. 06, 083 (2012).

[12] A. Salam and J. Strathdee, "On Kaluza-Klein Theory," Annals of Physics, 141, 316-352 (1982), pages 324-325, 346.

[13] N. Kim, T. Klose and J. Plefka, "Plane-wave Matrix Theory from $\mathcal{N} = 4$ Super Yang-Mills on $\mathbb{R} \times S^3$," Nucl.Phys. B671, 359 (2003) [hep-th/0306054], pages 4-6.

[14] G. 't Hooft, "Dimensional Reduction in Quantum Gravity," Essay dedicated to Abdus Salam, (gr-gc/9310006).

[15] L. Susskind, "The World as a Hologram," J. Math. Phys. 36:6377-6396 (1995) [arXiv:hep-th/9409089]

[16] C. Thorn, "In Sakharov Conference on Physics," Moscow, Vol 81, (1991)

[17] I. Klebanov, L. Susskind, Nucl. Phys. B309, 175 (1988)

[18] N. Beisert, C. Ahn, L.F. Alday, Z. Bajnok, J.M. Drummond, L. Freyhult, N. Gromov, R.A. Janik, V. Kazakov, T. Klose, G.P. Korchemsky, C. Kristjansen, M. Magro, T. McLoughlin, J.A. Minahan, R.I. Nepomechie, A. Rej, R. Roiban, S. Schafer-Nameki, C. Sieg, M. Staudacher, A. Torrielli, A.A. Tseytlin, P. Vieira, D. Volin, K. Zoubos, "Review of AdS/CFT Integrability: An Overview," Lett. Math. Phys. 99, 3 (2012) [arXiv:1012.3982]

[19] Griffiths, David. “Introduction to Quantum Mechanics (2nd ed.),” Prentice Hall. (2004). ISBN 0-13-111892-7.

[20] N. Beisert, “The $su(2|2)$ Dynamic S-Matrix,” Adv. Theor. Math. Phys. **12**, 948 (2008) doi:10.4310/ATMP.2008.v12.n5.a1 [hep-th/0511082].

[21] M. Staudacher, “The factorized S-matrix of CFT/AdS,” JHEP 0505, 054 (2005) [arXiv:hep-th/0412188].

[22] R. Roiban, “Magnon Bound-state Scattering in Gauge and String Theory,” JHEP 0704, 048 (2007) [arXiv:hep-th/0608049]

[23] M. Huszár, “Light Front Quantization by Dirac’s Method,” J. Phys. A: Math. Gen., Vol 9, No. 8, 1976.